Algebras of block Toeplitz matrices

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Consequence: The collection of Toeplitz operators contains two maximal algebras: analytic and coanalytic.

Toeplitz matrices

$\phi(z) = \sum_{-\infty}^{\infty} a_n z^n$. In the standard basis:

$$T_\phi = \begin{pmatrix} a_0 & a_1 & a_2 & \dots \\ a_{-1} & a_0 & a_1 & \ddots \\ a_{-2} & a_{-1} & a_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

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Again the collections of upper triangular and of lower triangular Toeplitz matrices are maximal algebras. There are others!

Circulants

Algebras of scalar Toeplitz matrices

Circulants

$$\left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_n & a_0 & a_1 & \ddots & \vdots \\ a_{n-1} & a_n & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_1 & \dots & \dots & a_n & a_0 \end{pmatrix}, a_i \in \mathbb{C} \right\}$$

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$$\mathfrak{A}_\alpha = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ \alpha a_n & a_0 & a_1 & \ddots & \vdots \\ \alpha a_{n-1} & \alpha a_n & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \alpha a_1 & \dots & \dots & \alpha a_n & a_0 \end{pmatrix}, a_i \in \mathbb{C} \right\}$$

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Particular cases:

1. for $\alpha = 0$, $\mathfrak{A}_0$ consists of the upper triangular Toeplitz matrices ($\alpha = \infty$: lower triangular Toeplitz matrices);
2. for $|\alpha| = 1$, $\mathfrak{A}_\alpha$ is the commutant of a unitary operator of multiplicity 1.

Block Toeplitz matrices

Problem (posed in Shalom, 1987)

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What are the maximal subalgebras of $\mathfrak{T}_{n,d}(\mathcal{A})$?

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3. Fix $\sigma + \tau = d$, $|\sigma - \tau| \leqslant 1$.

$$\mathcal{O}_{\sigma,\tau} = \left\{ \begin{pmatrix} \lambda I_\sigma & X \\ 0 & \lambda I_\tau \end{pmatrix} \middle| \lambda \in \mathbb{C}, X \in \mathcal{M}_{\sigma\times\tau}(\mathbb{C}) \right\}$$

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is a maximal commutative algebra (*Schur algebra*).
4. Suppose that M is a nonderogatory matrix; that is, its minimal polynomial is equal to its characteristic polynomial. Then the algebra $\mathcal{P}(M)$ generated by M is maximal commutative.

General facts

Lemma

Suppose A and B are block Toeplitz matrices, and the entries of A commute with the entries of B. Then the following are equivalent:

(i) *AB is a block Toeplitz matrix.*

(ii) *$AB = BA$.*

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Corollary

$\mathcal{S}$ is a set of commuting block Toeplitz matrices, whose entries all commute. Then all elements in the algebra generated by $\mathcal{S}$ are commuting block Toeplitz matrices.

Building blocks

Cyclic diagonal (of order k): $T = (T_{i-j})$, $T_{i-j} \neq 0$ only for $i - j = k \mod n$.

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A cyclic diagonal is characterized by two matrices A_k and A_{k-n}.

Theorem

1. *A maximal subalgebra of $\mathfrak{T}_{n,d}[\mathcal{A}]$ is generated as a linear subspace by the cyclic diagonals it contains.*
2. *The space of all pairs (A_k, A_{k-n}) is the same for different values of k.*

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Fix $A, B \in \mathcal{A} \subset \mathcal{M}_d$, with $\ker A \cap \ker B = \{0\}$.

Definition

$$\mathcal{F}_{A,B}^{\mathcal{A}} = \left\{ (T_{p-q})_{p,q=0}^{n-1} : T_j \in \mathcal{A}, AT_j = BT_{j-n}, j = 1, 2, \cdots n-1 \right\}.$$

Classes of algebras

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Theorem

- $\mathcal{F}^{\mathcal{A}}_{A,B}$ *is a maximal subalgebra of* $\mathfrak{T}_{n,d}(\mathcal{A})$.
- $\mathcal{F}^{\mathcal{A}}_{A,B} = \mathcal{F}^{\mathcal{A}}_{A',B'}$ *if and only if* $AB' = A'B$.

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Are these all?

A general result

Theorem

Suppose $\mathcal{B}$ is a maximal algebra of $\mathfrak{T}_{n,d}[\mathcal{A}]$ that contains at least one element with at least one off-diagonal invertible entry.
Then $\mathcal{B} = \mathcal{F}^{\mathcal{A}}_{A,B}$ for some A and B.

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Other maximal algebras: all nondiagonal terms noninvertible.

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- Any maximal algebra $\mathcal{B}$ of $\mathcal{T}_{n,d}$ with entries in $\mathcal{D}$ is equal to $\mathcal{F}^{\mathcal{D}}_{A,B}$ for some A and B.
- After reshuffling, we have

$$\mathcal{B} = \mathfrak{A}_{\alpha_1} \oplus \cdots \oplus \mathfrak{A}_{\alpha_d}.$$

$\mathcal{A} = \mathfrak{A}_0$

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Example

If $\mathcal{A} = \mathfrak{A}_0$ =upper triangular Toeplitz matrices of order d, then

- either $\mathcal{B} = \mathcal{F}^{\mathcal{A}}_{A,B}$ for some $A, B \in \mathfrak{A}_0$, $\ker A \cap \ker B = \{0\}$,
- or

$$\mathcal{B} = \mathfrak{B}_0 := \{T = (T_{p-q}) \in \mathcal{T}_{n,d}[\mathfrak{A}_0],\\ T_i \text{ noninvertible for all } i \neq 0\}.$$

$\mathfrak{B}_0$ is not of type $\mathcal{F}^{\mathcal{A}}_{A,B}$.

A classification

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Suppose the minimal polynomial of T is $p_T(X)$. Remember

$$q(T) \text{ invertible} \Leftrightarrow (q, p_T) = 1.$$

Particular case

Suppose $n = 2$, and T is nilpotent: $T^m = 0$, $T^{m-1} \neq 0$.

$$\mathfrak{T}_{2,d}(\mathcal{P}(T)) = \{ \begin{pmatrix} A_0 & A_1 \\ A_{-1} & A_0 \end{pmatrix} ; A_i \in \mathcal{P}(T) \}.$$

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Theorem

1. *All maximal subalgebras of* $\mathfrak{T}_{n,2}(\mathcal{P}(T))$ *are*

$$\mathrm{B}(r_+, r_-, \rho) = \{A = \begin{pmatrix} a_0(T) & T^{r_1} a_1(T) \\ T^{r_{-1}} a_{-1}(T) & a_0(T) \end{pmatrix} : \\ a_+(X) = \rho(x) a_-(X) \mod X^{m-r_+-r_-}\},$$

where $r_1, r_{-1} \in \mathbb{N}$, $r_1 + r_{-1} \leqslant m$, *and* $\rho \in \mathbb{C}[X]$ *is a polynomial with nonzero constant term.*

2. $\mathrm{B}(r_+, r_-, \rho) = \mathrm{B}(r'_+, r'_-, \rho')$
$\Leftrightarrow r_+ = r'_+, r_- = r'_-$, *and* $\rho(X) = \rho'(X) \mod X^{m-r_+-r_-}$.

General case

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Definition

Suppose $p_+, p_-, \rho \in \mathbb{C}[X]$ are three polynomials, such that p_+p_- divides p_T, while ρ and p_T are relatively prime; denote $q = p_T/p_+p_-$. We define $\mathfrak{B}(p_+, p_-, \rho)$ to be the set of matrices $A = (A_{i-j})_{i,j=0}^{n-1}$, where:

(a) $A_i \in \mathcal{P}(M)$ for all i with $-(n-1) \leqslant i \leqslant n-1$;

(b) $A_i = p_+(M)a_i(M)$, $A_{i-n} = p_-(M)a_{i-n}(M)$ for $i \geqslant 1$, where $a_i \in \mathbb{C}[X]$;

(c) $a_i = \rho a_{i-n} \mod q$ for all $i \geqslant 1$.

Theorem

These are all the maximal subalgebras of $\mathfrak{T}_{n,d}(\mathcal{P}(T))$.

Generalizations

u inner

- Model space and operator: $K_u = H^2 \ominus uH^2$., $S_u = P_{K_u} S | K_u$..
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Theorem (Sedlock, 2011)

All maximal algebras of truncated Toeplitz operators are $\mathfrak{B}_\alpha$ for $\alpha \in \mathbb{C} \cup \{\infty\}$, where

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Model spaces and operators

- $\Theta : \mathbb{D} \to \mathcal{L}(\mathbb{C}^d)$ is inner (isometric on $\mathbb{T}$);
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Theorem

1. $\{S_\Theta\}' = \{A_\Phi^\Theta : \Phi \in H^\infty(M_n(\mathbb{C}^d))\}$.
2. $\{S_\Theta\}'$ *is a maximal algebra of matrix valued TTOs.*

Question

What are other maximal algebras?

Thank you!