Von Neumann's inequality for operator-valued multishifts

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(Joint work with Rajeev Gupta and Shailesh Trivedi)

OTOA at ISI Bangalore

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Every pair of commuting contractions dilates to a pair of commuting unitaries.

Classical multishifts

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Let $T = (T_1, \ldots, T_d)$ be a contractive classical multishift with non-zero weights. Then T dilates to a d-tuple of commuting unitaries.

Operator-valued multishift

Let $\mathcal{H} = \oplus_{\alpha\in\mathbb{N}^d} H_\alpha$ be the orthogonal direct sum of H_α, $\alpha \in \mathbb{N}^d$.
An *operator-valued multishift* T on $\mathcal{H} = \oplus_{\alpha\in\mathbb{N}^d} H_\alpha$ with operator weights $\{A_\alpha^{(j)} : \alpha \in \mathbb{N}^d,\ j = 1, \ldots, d\}$ is a d-tuple of operators $T_1, \ldots, T_d$ in $\mathcal{H}$ defined by

$$D(T_j) := \Big\{x = \oplus_{\alpha\in\mathbb{N}^d} x_\alpha \in \mathcal{H} : \sum_{\alpha\in\mathbb{N}^d} \|A_\alpha^{(j)} x_\alpha\|^2 < \infty\Big\},$$

$$T_j(\oplus_{\alpha\in\mathbb{N}^d} x_\alpha) = \oplus_{\alpha\in\mathbb{N}^d} A_{\alpha-\varepsilon_j}^{(j)} x_{\alpha-\varepsilon_j}, \quad x = \oplus_{\alpha\in\mathbb{N}^d} x_\alpha \in D(T_j),\ j = 1, \ldots, d.$$

For each $\alpha \in \mathbb{N}^d$ and $j = 1, \ldots, d$, if $\alpha_j = 0$, then we interpret $A_{\alpha-\varepsilon_j}^{(j)}$ to be a zero operator and $x_{\alpha-\varepsilon_j}$ as a zero vector.

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- For $j = 1, \ldots, d$, T_j is bounded if and only if $\sup_{\alpha \in \mathbb{N}^d} \|A_\alpha^{(j)}\| < \infty$.
- For $i, j = 1, \ldots, d$, T_i commutes with T_j if and only if $A_{\alpha+\varepsilon_j}^{(i)} A_\alpha^{(j)} = A_{\alpha+\varepsilon_i}^{(j)} A_\alpha^{(i)}$ for all $\alpha \in \mathbb{N}^d$.

Operator-valued multishift

If $H_\alpha = H$ for all $\alpha \in \mathbb{N}^d$, then we denote $\mathcal{H} = \oplus_{\alpha \in \mathbb{N}^d} H$ by $\ell^2_H(\mathbb{N}^d)$.

Theorem

Let d be a positive integer and H be a complex Hilbert space. Let $T = (T_1, \ldots, T_d)$ be a commuting operator-valued multishift on $\ell^2_H(\mathbb{N}^d)$ with unitary operator weights $\{A^{(j)}_\alpha : \alpha \in \mathbb{N}^d,\ j = 1, \ldots, d\}$. Then T satisfies the von Neumann's inequality.

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Any two commuting operator-valued multishifts with unitary operator weights are unitarily equivalent.

Von Neumann's inequality for tensor product of tuples

Theorem (R. Gupta, —, S. Trivedi)

Let d be a positive integer and let $A = (A_1, \ldots, A_d)$, $B = (B_1, \ldots, B_d)$ be two commuting d-tuples of contractions on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. Suppose that B satisfies the matrix-version of von Neumann's inequality and $(1, \ldots, 1)$ belongs to the algebraic spectrum $\sigma(B)$ of B. Then $A \otimes B = (A_1 \otimes B_1, \ldots, A_d \otimes B_d)$ satisfies the von Neumann's inequality if and only if A satisfies the von Neumann's inequality.

Operator-valued multishift and the von Neumann's inequality

Corollary

Let H be a complex Hilbert space and d be a positive integer. Suppose that $A = (A_1, \dots, A_d)$ is a d-tuple of commuting contractions on H and $T = (T_1, \dots, T_d)$ is the commuting operator-valued multishift on $\ell^2_H(\mathbb{N}^d)$ with operator weights given by $A^{(j)}_\alpha = A_j$ for all $\alpha \in \mathbb{N}^d$ and $j = 1, \dots, d$. Then T satisfies the von Neumann's inequality if and only if A satisfies the von Neumann's inequality.

Proof of main result

- $p(z) = \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \leq k}} a_\alpha z^\alpha, \quad a_\alpha \in \mathbb{C},\ z \in \mathbb{C}^d$, be a polynomial.
- There exists a multiplicative linear functional χ on the unital Banach algebra $\mathfrak{B}$ generated by $B_1, \ldots, B_d$ such that $\chi(B_j) = 1$ for all $j = 1, \ldots, d$.

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- By Hahn-Banach theorem there exists a contractive linear functional ϕ on $\mathcal{B}(\mathcal{H})$ such that $\phi(p(A)) = \|p(A)\|$. Then $\phi \otimes \chi$ is a contractive linear functional on $\mathcal{B}(\mathcal{H}) \otimes \mathfrak{B}$.

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- $\|p(A \otimes B)\| \geq |(\phi \otimes \chi)p(A \otimes B)| = \left| \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \leq k}} a_\alpha \phi(A^\alpha)\chi(B^\alpha) \right| = |\phi(p(A))| = \|p(A)\|$.

This establishes that A satisfies the von Neumann's inequality if $A \otimes B$ satisfies the von Neumann's inequality.

Proof of main result

- Consider the polynomial $p(z, w)$ given by
  $p(z,w) = \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|\le k}} a_\alpha z^\alpha w^\alpha, \quad z, w \in \mathbb{C}^d$.
- For each fixed $w \in \mathbb{C}^d$, we get
  $\|p(A,w)\| = \left\| \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|\le k}} a_\alpha A^\alpha w^\alpha \right\| \le \sup_{z\in\mathbb{D}^d} |p(z,w)|$.

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- For each fixed $w \in \mathbb{C}^d$, we get $\|p(A, w)\| = \left\| \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \le k}} a_\alpha A^\alpha w^\alpha \right\| \le \sup_{z \in \mathbb{D}^d} |p(z, w)|$.
- Define the (matrix-valued) polynomial $p_{F}(A, w) = \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \le k}} a_\alpha \, P_F A^\alpha P_F \, w^\alpha, \quad w \in \mathbb{C}^d$.
- Since B satisfies the matrix version of von Neumann's inequality, $\|p_{F}(A, B)\| \le \sup_{w \in \mathbb{D}^d} \|p_{F}(A, w)\| \le \sup_{w \in \mathbb{D}^d} \|p(A, w)\|$.

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- For each fixed $w\in\mathbb{C}^d$, we get $\|p(A,w)\| = \left\| \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|\le k}} a_\alpha A^\alpha w^\alpha \right\| \le \sup_{z\in\mathbb{D}^d} |p(z,w)|.$
- Define the (matrix-valued) polynomial $p_{_F}(A,w) = \sum_{\substack{\alpha\in\mathbb{N}^d \\ |\alpha|\le k}} a_\alpha\, P_F A^\alpha P_F\, w^\alpha, \quad w\in\mathbb{C}^d.$
- Since B satisfies the matrix version of von Neumann's inequality, $\|p_{_F}(A,B)\| \le \sup_{w\in\mathbb{D}^d} \|p_{_F}(A,w)\| \le \sup_{w\in\mathbb{D}^d} \|p(A,w)\|.$
- Now it follows that $\|p(A\otimes B)\| \le \sup_{w\in\mathbb{D}^d} \|p(A,w)\| \le \sup_{w\in\mathbb{D}^d} \sup_{z\in\mathbb{D}^d} |p(z,w)| = \sup_{z\in\mathbb{D}^d} |p(z)|.$

This completes the proof of the theorem.

Example

Let $c \in (0, 1/(6+\sqrt{30}))$, consider the Varopoulos operators on $\mathbb{C}^4$ given by

$$V_j = \begin{pmatrix} 0 & x_j & y_j & 0 \\ 0 & 0 & 0 & x_j \\ 0 & 0 & 0 & y_j \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad j = 1, 2, 3,$$

where $x_j, y_j \in \mathbb{R}$ and $x_j^2 + y_j^2 = (1-c)^2$ for each $j = 1, 2, 3$. Set $A_j := cI + V_j$ for all $j = 1, 2, 3$.

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Let $T = (T_1, T_2, T_3)$ be the commuting operator-valued multishift on $\ell^2_{\mathbb{C}^4}(\mathbb{N}^d)$ with operator weights given by $A_\alpha^{(j)} = A_j$ for all $\alpha \in \mathbb{N}^d$ and $j = 1, 2, 3$.

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Now, consider the Varopoulos-Kaijser polynomial

$$p_v(z_1, z_2, z_3) := z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_2z_3 - 2z_3z_1.$$

It is shown that $\sup_{z \in \mathbb{D}^3} |p_v(z)| = 5$. It can be concluded that $\|p_v(A_1, A_2, A_3)\| \geq 6(1-c)^2 > 5$.

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