On a presentation of the spin planar algebra

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Introduction to planar algebra

Definition (Shaded planar tangle)

A (shaded) planar tangle is the data of finitely many input disks, one output disk, non-intersecting strings giving an even number, say $2n$ intervals per disk and one $\star$ -marked interval per disk. There is given a checkerboard shading of the regions (connected components of the complement of the curves) such that across any curve, the shading toggles. A disc with $2n$ marked points on its boundary is said to be an $(n,+)$ disc or an $(n,-)$ disc according as its $\star$ -arc is adjacent to a white or a black region. The pair (n,ϵ) is called the colour of a disk.The colour of a tangle is the colour of its external disc. The tangles are defined upto isotopy of $\mathbb{R}^2$.

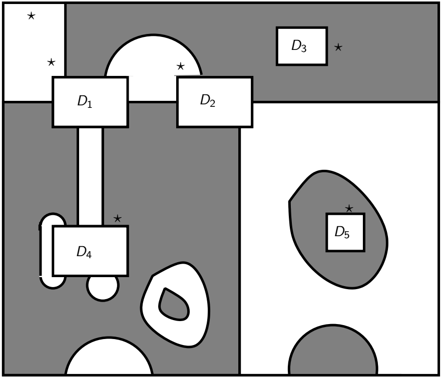


Figure 1: Example of a $(4, +)$ tangle.

Two basic operations can be performed on tangles to produce new tangles from the old ones.

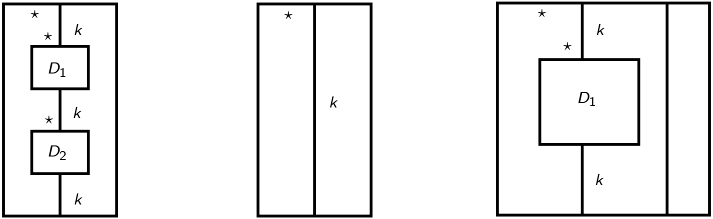
1. Renumbering of the disks
2. Substitution

Definition (Planar algebra)

A planar algebra is a family of vector spaces $P = (P_{(n,\pm)})_{n \in \mathbb{N} \cup \{0\}}$ called n -box spaces, on which the planar tangle acts as multilinear maps. That is if $T = T^{(k_0,\epsilon_0)}_{(k_1,\epsilon_1),(k_2,\epsilon_2),\ldots,(k_b,\epsilon_b)}$ is a planar tangle with b input discs of colours $(k_1,\epsilon_1),(k_2,\epsilon_2),\ldots,(k_b,\epsilon_b)$ and output disc of colour (k_0,ϵ_0), then T gives a multilinear map Z_T from $P_{(k_1,\epsilon_1)} \times P_{(k_2,\epsilon_2)} \times \ldots \times P_{(k_b,\epsilon_b)} \to P_{(k_0,\epsilon_0)}$ satisfying some compatibility conditions under renumbering and substitution of tangles.

Proposition

Let P be a planar algebra. Then for every colour $(k, \pm)$, the vector space $P_{(k,\pm)}$ has the natural structure of an associative unital algebra with the multiplication specified by the $(k, \pm)$-multiplication tangle $M_{(k,\pm),(k,\pm)}^{(k,\pm)}$ and the unit given by $Z_{U_{(k,\pm)}^{(k,\pm)}}(1_{\mathbb{C}})$ where $U_{(k,\pm)}^{(k,\pm)}$ is the unit tangle. Furthermore, inclusion tangles induce homomorphisms of unital algebras.



$M^{(k,\pm)}_{(k,\pm),(k,\pm)}$ $\quad$ $U^{(k,\pm)}_{(k,\pm)}$ $\quad$ $I^{(k+1,\pm)}_{(k,\pm)}$

Figure 2: Multiplication, unit and inclusion tangles

A planar algebra $P = \{P_{(k,\epsilon)}\}$ is finite dimensional if each space $P_{(k,\epsilon)}$ is finite dimensional.

Definition ($*$- planar algebra)

A $*$- planar algebra is a planar algebra P with each $P_{(k,\epsilon)}$ equipped with an involution map $*$ such that for any tangle T with b input discs with colours $(k_1, \epsilon_1), (k_2, \epsilon_2), \ldots (k_b, \epsilon_b)$ and output disc of colour (k_0, ϵ_0), $(Z_T(x_1 \otimes \ldots \otimes x_n))^* = Z_{T^*}(x_1^* \otimes \ldots \otimes x_n^*)$

Definition (C^*- planar algebra)

A $*$-planar algebra P is said to be a C^*-planar algebra if there exist positive normalized traces $\tau_\pm : P_{(0,\pm)} \to \mathbb{C}$ such that all the traces $\tau_\pm \circ Z_{TR^{(0,\pm)}}$ defined on $P_{(k,\pm)}$ are faithful and positive.

Presentation of a planar algebra

We can have a presentation for a planar algebra just like we have presentations of a group. The approach is also similar to the free group construction by a label set L and a set of relations R.

Definition (Universal planar algebra on the label set L)

Let $L = \amalg_{(k,\epsilon)\in Col} L_{(k,\epsilon)}$ be a disjoint union of an arbitrary collection $L_{(k,\epsilon)}$ of 'label sets'- where some $L_{(k,\epsilon)}$ may be empty. Define a (k_0, ϵ_0)- tangle labeled by L to be a (k_0, ϵ_0) tangle T as above subject to one additional constraint- viz.,that T is allowed to have an internal disc of colour (m, ϵ) only if $L_{(m,\epsilon)} \neq \emptyset$ - and equipped with the extra structure of a label from $L_{(m,\epsilon)}$ associated to every internal disc of colour (m, ϵ). Define $P_{(k,\epsilon)}(L)$ to be the vector space with basis given by the collection of '(k, ϵ)- tangles labeled by L' and let $P(L) = \{P_{(k,\epsilon)}(L) : (k, \epsilon) \in Col\}$. Then $P(L)$ is called the universal planar algebra defined on the label set L.

Definition (Planar ideal)

A family $J = \{J_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ is said to be a *planar ideal* of a planar algebra $P = \{P_{(k,\epsilon):(k,\epsilon)\in Col}\}$ if

1. $J_{(k,\epsilon)}$ is a vector subspace of $P_{(k,\epsilon)}$ for each (k,ϵ) and
2. if T is any (k_0,ϵ_0) tangle, with b internal discs D_i of colour (k_i,ϵ_i) for $1 \leq i \leq b$, then it is demanded that $Z_T(\otimes_{i=1}^{b} x_i) \in J_{(k_0,\epsilon_0)}$ whenever $x_i \in J_{(k_i,\epsilon_i)}$ for any one i.

If J is a planar ideal of a planar algebra P, then $P/J = \{P_{(k,\epsilon)}/J_{(k,\epsilon)} : k,\epsilon) \in Col\}$ is naturally a planar algebra and that the natural quotient maps define a ‘morphism of planar algebras’.

Definition (Presentation of a planar algebra)

Given an arbitrary label set $L = \amalg_{(k,\epsilon)\in Col} L_{(k,\epsilon)}$, and an arbitrary subset $R = \{R_{(k,\epsilon)} : (k,\epsilon) \in Col\}$ of the universal planar algebra $P(L)$ with generating set L let $J(R)$ be the smallest ideal containing R, and define $P(L;R)$ to be the quotient $P(L)/J(R)$.
An arbitrary planar algebra Q will be said to be 'presented on the generating set L with the relations R if there exists an isomorphism $\Phi : P(L;R) \to Q$ of planar algebras; and we shall say that Q is 'presented by the map Φ.

Spin planar algebra

The spin planar algebra P is well known from the very first paper of Jones on planar algebra [Jns1999]. It can be identified with the planar algebra associated to a specific bipartite graph (see Example 4.2 of [Jns2000]) Γ given below.

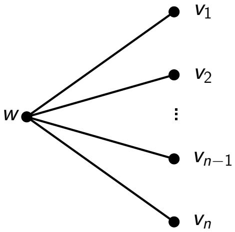


Figure 3: The bipartite graph Γ

The algebra structure of $P_{(k,\pm)}$ are as follows where the inclusion can be explicitly calculated (say for the parameter $n = 2$):

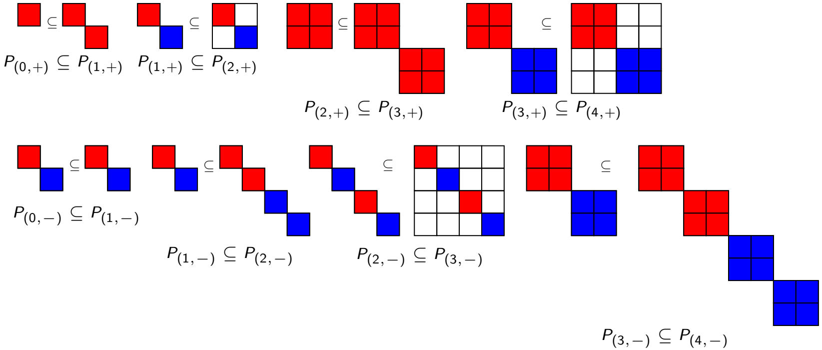


Figure 4: Inclusion of the spaces $P_{(k,\pm)} \subseteq P_{(k+1,\pm)}$, for $k = 0, 1, 2, 3$

Presentation of the spin planar algebra - construction

Let $S = \{s_1, s_2, \cdots, s_n\}$ be a finite set. We will define a certain abstract planar algebra over $\mathbb{C}$ associated to this set. Begin with the label set $L = L_{(0,-)} = S$ equipped with the identity involution $*$. Consider the quotient $P = P(L, R)$ of the universal planar algebra $P(L)$ by the set R of relations in Figures 5 and 6 (where δ_{ij} denotes the Kronecker delta).

Figure 5: The white and black modulus relations

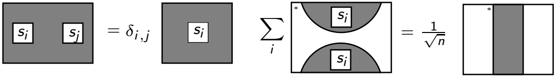


Figure 6: The multiplication and the black channel relations

Main theorem

The below theorem establishes various structural properties of the abstract planar algebra P defined above.

Theorem

With the notations above, the planar algebra P is a finite dimensional C^*-planar algebra with modulus $\sqrt{n}$ and such that $dim(P_{(0,+)}) = 1$ and $dim(P_{(0,-)}) = n$. For $k > 0$, $dim(P_{(k,\pm)}) = n^k$.

Isomorphism of P with $P(\Gamma)$

We know that for a connected bipartite graph Γ with vertex set $V = V_+ \amalg V_-$ and edge set E, the planar algebra $P(\Gamma)$ of the bipartite graph has vector spaces given by $P(\Gamma)_{(k,\pm)}$ being the vector space with basis all loops of lengt $2k$ in Γ based at a vertex in $V_\pm$.The details of the construction can be found in [Jns2000]. The proposition below identifies P with $P(\Gamma)$ for the bipartite graph Γ above(see figure 3).

Proposition

With $S = \{v_1, \dots, v_n\}$, the planar algebra P constructed in the main theorem is isomorphic to $P(\Gamma)$ by the map that takes $v_i \in P_{(0,-)}$ to the loop of length 0 based at v_i in $P(\Gamma)_{(0,-)}$.

V. F. R. Jones, *Planar algebras*, To appear in New Zealand J. Math. arXiv:math/9909027.

V. F. R. Jones, *The planar algebra of a bipartite graph*, Knots in Hellas '98 (Delphi), Ser. Knots Everything Vol. 24, 94–117, 2000.

Vijay Kodiyalam, Sruthymurali, Sohan Lal Saini and V S Sunder, *On a Presentation of the Spin planar algebra*, accepted

Thank You!