Rank of Co-Doubly Commuting Hilbert Modules

(Joint work with Arup Chattopadhyay and Jaydeb Sarkar)

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Rank and Wandering subspaces.

Let $T := (T_1, \ldots, T_n)$ be an n-tuple of commuting bounded linear operators on a Hilbert space $\mathcal{H}$, and let E be a non-empty subset of $\mathcal{H}$. The T-generating hull of E is defined by

$$[E]_T = \bigvee_{\boldsymbol{k} \in \mathbb{N}^n} T^{\boldsymbol{k}}(E).$$

Then the *rank* of T is the unique number

$$\text{rank}(T) = \min\{\#E : [E]_T = \mathcal{H}, E \subseteq \mathcal{H}\}.$$

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Let $\mathcal{S}$ be a closed T-invariant subspace of $\mathcal{H}$, then

$$\text{rank}\,(\mathcal{S}) := \text{rank}(T|_{\mathcal{S}}).$$

A closed T-invariant subspace $\mathcal{S} \subseteq \mathcal{H}$ is said to have the *wandering subspace property* with respect to $T|_{\mathcal{S}}$ if

$$\mathcal{S} = \bigvee_{\boldsymbol{k} \in \mathbb{N}^n} T^{\boldsymbol{k}}(\mathcal{W}_T(\mathcal{S})); \quad \mathcal{W}_T(\mathcal{S}) := \mathcal{S} \ominus \big(\sum_{l=1}^{n} T_l|_{\mathcal{S}}(\mathcal{S})\big).$$

Rank in Hilbert Spaces.

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Fact: Let $\mathcal{S}$ be a T-invariant subspace of a Hilbert space $\mathcal{H}$ then

$$\text{rank}\ (T|_{\mathcal{S}}) \geq \dim\ (\mathcal{W}_T(\mathcal{S})).$$

If rank $(T|_{\mathcal{S}}) < \infty$ then rank $(T|_{\mathcal{S}}) = \dim\,(\mathcal{W}_T(\mathcal{S}))$ if and only $\mathcal{S}$ has the *wandering subspace property* with respect to $T|_{\mathcal{S}}$.

Reproducing kernel Hilbert spaces and modules

- Let $\mathcal{E}$ be a Hilbert space, Λ be a set and $K : \Lambda \times \Lambda \to \mathcal{B}(\mathcal{E})$ be a function. Let $\mathcal{H}_K$ be a Hilbert space of $\mathcal{E}$-valued functions on Λ. Then $\mathcal{H}_K$ is said to be a reproducing kernel Hilbert space if

$$\langle f, K_\lambda \eta \rangle_{\mathcal{H}_K} = \langle f(\lambda), \eta \rangle_{\mathcal{E}}, \, \lambda \in \Lambda, \eta \in \mathcal{E}.$$

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- Let Ω be a bounded domain in $\mathbb{C}^n$ and let K be a $\mathcal{B}(\mathcal{E})$-valued kernel on Ω. Let $K(\boldsymbol{z}, \boldsymbol{w})$ be holomorphic in $\{z_1, \ldots, z_n\}$. Then

$$\mathcal{H}_K = \overline{\text{span}}\{K(\cdot, \boldsymbol{w})\eta : \boldsymbol{w} \in \Omega, \eta \in \mathcal{E}\} \subseteq \mathcal{O}(\Omega, \mathcal{E}).$$

We say that $\mathcal{H}_K$ is a *reproducing kernel Hilbert module* if

$$z_j \mathcal{H}_K \subseteq \mathcal{H}_K \qquad (j = 1, \ldots, n).$$

$$(M_{z_j} f)(\boldsymbol{w}) = w_j f(\boldsymbol{w}) \qquad (\boldsymbol{w} \in \Omega, f \in \mathcal{H}_K),$$

induces a $\mathbb{C}[\boldsymbol{z}]$-module action on $\mathcal{H}_K$ as follows:

$$p \cdot h = p(M_{z_1}, \ldots, M_{z_n})h \qquad (p \in \mathbb{C}[z_1, \ldots, z_n], h \in \mathcal{H}_K).$$

Setup

Let $\{\mathcal{H}_{K_i}\}_{i=1}^n$ be a collection of reproducing kernel Hilbert modules over $\mathbb{D}$ corresponding to the positive definite kernel functions $K_i : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$, $i = 1, \ldots, n$. Thus

$$K(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^{n} K_i(z_i, w_i), \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n)$$

defines a positive definite kernel on $\mathbb{D}^n$.

$$\mathcal{H}_K \cong \mathcal{H}_{K_1} \otimes \cdots \otimes \mathcal{H}_{K_n}$$

is a reproducing kernel Hilbert module over $\mathbb{C}[\boldsymbol{z}]$.

Analytic RKHM

Definition

$\mathcal{H}_K$ is said to be an analytic reproducing kernel Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ if it satisfies the following conditions (for $i = 1, \ldots, n$):

- $1 \in \mathcal{H}_{K_i}$,
- *K_i^{-1} is a polynomial in z and $\bar{w}$,*
- *There does not exist two non-zero quotient modules of $\mathcal{H}_{K_i}$ which are orthogonal to each other.*

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Examples:

(i) $H^2(\mathbb{D}^n)$,

(ii) $L^2_{a,\alpha}(\mathbb{D}^n)$.

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A quotient module $\mathcal{Q}$ of $\mathcal{H}_K$ is *doubly commuting* if for $1 \leq i < j \leq n$,

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A submodule $\mathcal{S}$ of $\mathcal{H}_K$ is *co-doubly commuting* if the quotient module $\mathcal{H}_K/\mathcal{S}$ is doubly commuting.

Characterization

Theorem (Chattopadhyay, Das and Sarkar '2015)

Let $\mathcal{H}_K = \mathcal{H}_{K_1} \otimes \ldots \otimes \mathcal{H}_{K_n}$ be an analytic Hilbert module over $\mathbb{C}[\boldsymbol{z}]$ and $\mathcal{S}$ be a submodule of $\mathcal{H}_K$. Then $\mathcal{S}$ is co-doubly commuting if and only if

$$\mathcal{S} = (\mathcal{Q}_1 \otimes \ldots \otimes \mathcal{Q}_n)^{\perp} = \sum_{i=1}^{n} \mathcal{H}_{K_1} \otimes \ldots \mathcal{H}_{K_{i-1}} \otimes \mathcal{Q}_i^{\perp} \otimes \mathcal{H}_{K_{i+1}} \otimes \ldots \otimes \mathcal{H}_{K_n},$$

for some quotient modules $\mathcal{Q}_i$ of $\mathcal{H}_{K_i}$ for $i = 1, \ldots, n$.

Rank in Hardy space over polydisc

Question (Douglas, Yang '2000)

Let $\mathcal{S}$ be a rank one co-doubly commuting submodule of $H^2(\mathbb{D}^2)$, then $\mathcal{S} = \Theta H^2(\mathbb{D}^2)$ for some inner function Θ depending on a single variable?

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Let $\mathcal{S}$ be a co-doubly commuting submodule of $H^2(\mathbb{D}^2)$, then

$$\mathcal{S} = (\mathcal{Q}_\phi \otimes \mathcal{Q}_\psi)^\perp,$$

for some inner functions $\phi, \psi \in H^\infty(\mathbb{D})$.

Idea of the proof

Lemma

Let $T = (T_1, \ldots, T_n)$ be a n-tuple of commuting operators on Hilbert space $\mathcal{H}$. Let $\mathcal{S}_1, \mathcal{S}_2$ be two joint T-invariant subspaces of $\mathcal{H}$ and $\mathcal{S}_2 \subseteq \mathcal{S}_1$. If $\mathcal{S} = \mathcal{S}_1 \ominus \mathcal{S}_2$ then,

$$\text{rank}\,(P_{\mathcal{S}} T|_{\mathcal{S}}) \leq \text{rank}\,(T|_{\mathcal{S}_1}).$$

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$$\mathcal{S} = (\mathcal{S}_\phi \otimes H^2(\mathbb{D})) + (H^2(\mathbb{D}) \otimes \mathcal{S}_\psi)$$

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Genralization.

Question

Let $n \geq 2$ and let $\{\phi_j\}_{j=1}^n \subseteq H^\infty(\mathbb{D})$ be inner functions. Is then

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(i) For $\boldsymbol{k} \in \mathbb{N}^2$,

$$\begin{aligned} \left(M_{\boldsymbol{z}}^{\boldsymbol{k}}(\mathcal{S}_\phi \otimes \mathcal{Q}_\psi)\right) \quad &\perp \quad (\mathcal{Q}_\phi \otimes \mathcal{S}_\psi), \\ \left(M_{\boldsymbol{z}}^{\boldsymbol{k}}(\mathcal{Q}_\phi \otimes \mathcal{S}_\psi)\right) \quad &\perp \quad (\mathcal{S}_\phi \otimes \mathcal{Q}_\psi). \end{aligned}$$

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$$\mathcal{S}_\phi \otimes \mathcal{Q}_\psi = \bigvee_{\boldsymbol{k} \in \mathbb{N}^n} (P_{\mathcal{S}_\phi \otimes \mathcal{Q}_\psi} M_{\boldsymbol{z}}|_{\mathcal{S}_\phi \otimes \mathcal{Q}_\psi})^{\boldsymbol{k}} (\mathcal{W}_{P_{\mathcal{S}_\phi \otimes \mathcal{Q}_\psi} M_{\boldsymbol{z}}|_{\mathcal{S}_\phi \otimes \mathcal{Q}_\psi}} (\mathcal{S}_\phi \otimes \mathcal{Q}_\psi))$$

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Additive formula for muliplicities.

Let $T = (T_1, \ldots, T_n)$ be a n-tuple of bounded operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ such that rank$(P_{\mathcal{S}} T|_{\mathcal{S}}) < \infty$.

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- Suppose, $\mathcal{S}$ can be orthogonaly decomposed into the sum $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$, such that,

$$T^{\boldsymbol{k}}(\mathcal{S}_1) \perp \mathcal{S}_2 \text{ and } T^{\boldsymbol{k}}(\mathcal{S}_2) \perp \mathcal{S}_1, \quad (\boldsymbol{k} \in \mathbb{N}^n). \tag{1}$$

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- $\mathcal{S}_i = \bigvee_{\boldsymbol{k} \in \mathbb{N}^n} (P_{\mathcal{S}_i} T|_{\mathcal{S}_i})^{\boldsymbol{k}} \mathcal{W}_{(P_{\mathcal{S}_i} T|_{\mathcal{S}_i})}(\mathcal{S}_i)$ that is $\mathcal{S}_i$ has the wandering subspace property with respect to $P_{\mathcal{S}_i} T|_{\mathcal{S}_i}$.

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Theorem (Chattopadhay, Sarkar, S-)

If rank $(P_{\mathcal{S}_1} T|_{\mathcal{S}_1}) = m$ *and rank* $(P_{\mathcal{S}_2} T|_{\mathcal{S}_2}) = n$ *then*

$$\textit{rank}\,(P_{\mathcal{S}} T|_{\mathcal{S}}) = m + n.$$

Proposition (Chattopadhay, Sarkar, S-)

Let $\mathcal{H}_i$ be an analytic reproducing kernel Hilbert module for $i = 1, \ldots, n$ and $\mathcal{S}$ be a co-doubly commuting submodule of $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$ with finite rank, that is,

$$\mathcal{S} = \sum_{i=1}^{n} \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_{i-1} \otimes \mathcal{S}_i \otimes \mathcal{H}_{i+1} \otimes \ldots \otimes \mathcal{H}_n; \quad \text{rank}\,(M_{\boldsymbol{z}}|_{\mathcal{S}}) < \infty,$$

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where $\mathcal{S}_i$ is a M_z-invariant subspace of $\mathcal{H}_i$. If each $\mathcal{S}_i$ has the wandering subspace property with respect to M_z,

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where $\mathcal{S}_i$ is a M_z-invariant subspace of $\mathcal{H}_i$. Let $\mathcal{E}$ be a closed subspace of $\mathcal{S}$ defined by,

$$\mathcal{E} = (\mathcal{S}_1 \otimes \mathcal{Q}_2 \otimes \ldots \otimes \mathcal{Q}_n) \oplus (\mathcal{Q}_1 \otimes \mathcal{S}_2 \otimes \mathcal{Q}_3 \otimes \ldots \otimes \mathcal{Q}_n) \oplus \ldots \oplus (\mathcal{Q}_1 \otimes \mathcal{Q}_2 \otimes \ldots \otimes \mathcal{S}_n).$$

Then,

$$\text{rank}\,(P_{\mathcal{E}} M_{\mathbf{z}}|_{\mathcal{E}}) \leq \text{rank}\,(M_{\mathbf{z}}|_{\mathcal{S}}).$$

Theorem (Chattopadhay, Sarkar, S-)

With the same hypothesis as before,

$$\mathit{rank}\,(\mathcal{S}) = \mathit{rank}\,(M_{\mathbf{z}}|_{\mathcal{S}}) = \sum_{k=1}^{n} \mathit{rank}\,(M_z|_{\mathcal{S}_k}).$$

Applications:

Theorem (Chattopadhay, Sarkar, S-)

Let $\mathcal{S} = (\mathcal{Q}_1 \otimes \ldots \otimes \mathcal{Q}_n)^{\perp}$ *be a co-doubly commuting submodule of* $H^2(\mathbb{D}^n)$*. Then the rank of* $\mathcal{S}$ *is equal to the number of non-zero quotient modules* $\mathcal{Q}_i$ *which are different from* $H^2(\mathbb{D})$ *.*

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Corollary

Let $\mathcal{S}$ be a co-doubly commuting submodule of $H^2(\mathbb{D}^n)$. Then $rank\,(\mathcal{S}) = m$ implies $\mathcal{S} = \Theta H^2(\mathbb{D}^n)$ for some n-m variables inner function $\Theta \in H^\infty(\mathbb{D}^{n-m})$.

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Thank You!