A class of Sub-Hardy Hilbert Spaces Associated with Weighted Shifts

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Joint work with **Dinesh Singh**

Outline of the talk

- History and Motivation
- Some Notations and Definitions
- Statement of our Main Result
- Analogue of Wold's Decomposition
- Sketch of the proof of the main result
- Important consequences

History and Motivation

- Arne Beurling (1949) - Characterizes the closed subspaces of H^2 that are invariant under the action of T_z, the operator of multiplication with the coordinate function z.
- Peter Lax (1959)- Vector-valued generalization of Beurlings's work for shifts of finite multiplicity.
- Paul Halmos (1961)- Vector-valued generalization of Beurlings's work for shifts of infinite multiplicity.
- Louis de Branges - Not only extended Beurling's theorem but also its vector-valued generalizations due to Lax and Halmos.
- U. N. Singh and D. Singh (1991)- Generalized de Branges theorem (scalar case).

Notations and Definitions

- H^2- the class of analytic function on $\mathbb{D}$ whose Taylor coefficients are square summable.
  - (i) H^2 is a Hilbert space with respect to the inner product

    $$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

    for $f = \sum_{n=0}^{\infty} a_n z^n$ and $g = \sum_{n=0}^{\infty} b_n z^n$ in H^2.
  - (ii) $\{z^n\}_{n=0}^{\infty}$ forms an orthonormal basis for H^2.

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- H^∞-the class of bounded analytic functions on $\mathbb{D}$.
  (i) H^∞ is a Banach algebra with $||\phi||_\infty = sup\{|\phi(z)| : z \in \mathbb{D}\}$.
  (ii) $H^\infty = \{\phi \in H^2 : \phi H^2 \subseteq H^2\}$.

Notations and Definition contd...

- Let $\{\beta_n\}$ be a sequence of positive numbers.

$$H^2(\beta) = \left\{ f(z) = \sum_{n=0}^{\infty} \alpha_n z^n : \sum_{n=0}^{\infty} |\alpha_n|^2 \beta_n^2 < \infty \right\}$$

with the inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \alpha_n \overline{\gamma_n} \beta_n^2$$

for all $f = \sum_{n=0}^{\infty} \alpha_n z^n$ and $g = \sum_{n=0}^{\infty} \gamma_n z^n$ in $H^2(\beta)$.

$H^2(\beta)$ is a Hilbert space with respect to the above inner product space.

For $\beta_n = 1$ for all n, $H^2(\beta) = H^2$.

Notations and Definition contd...

- $T \in \mathcal{B}(H)$ is called an injective weighted shift with weight sequence $\{w_n\}_{n=0}^{\infty}$ if

$$Te_n = w_n e_{n+1},$$

where $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis for H and $\{w_n\}_{n=0}^{\infty}$ is a bounded sequence of positive numbers.

When $H = H^2$, $e_n = z^n$ and $w_n = 1$, we use T_z to denote the injective weighted shift operator.

- $T \in \mathcal{B}(H)$ is said to shift an orthogonal basis $\{h_n\}$ of H if $Th_n = h_{n+1}$ for each n.

- **A. Beurling**: If M is a closed subspace of H^2 invariant under the action of T_z, then there is an inner function b (i.e., $|b| = 1$ a.e. on $\mathbb{T}$) such that $M = bH^2$.

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- **de Branges**: Let M be a Hilbert space such that:
  (i) M is contractively contained in H^2, that is, $M \subseteq H^2$ and $||f||_2 \leq ||f||_M$,
  (ii) $T_z(M) \subseteq M$ and T_z is an isometry on M.

  Then there exists a $b \in H^\infty$ with $||b||_\infty \leq 1$ such that

$$M = bH^2 \quad and \quad ||bf||_M = ||f||_2 \; \forall f \in H^2$$

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- **Singh and Singh**: Let M be a Hilbert space such that:
  (i) $M \subseteq H^2$,
  (ii) $T_z(M) \subseteq M$ and T_z acts isometrically on M.

  Then there exists a $b \in H^\infty$ such that

$$M = bH^2 \quad and \quad ||bf||_M = ||f||_2 \; \forall f \in H^2.$$

Question

Can we weaken the hypotheses any further?

Our Theorem

Theorem (L. & Singh)

Let M be a Hilbert space contained in H^2. Suppose the operator T_z, which denotes multiplication by z, is well defined on M, and satisfies:

(i) *There exists a $\delta > 0$ such that $\delta||f||_M \leq ||T_z f||_M \leq ||f||_M$ for all $f \in M$.*

(ii) *For each $n \in \mathbb{N}$, $T_z^{*n} T_z^{n+1}(M) \subseteq T_z(M)$ (the adjoint of T_z is with respect to the inner product on M).*

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Then T_z acts as a weighted shift on M, and there exists a $b \in H^\infty$ such that

$$M = \overline{bH^2} \quad (\text{the closure is in the norm of } M)$$

and

$$||bf||_M \leq ||f||_2 \quad \text{for all } f \in H^2.$$

Analogue of Wold's decomposition

Lemma (L. & Singh)

Let $T \in \mathcal{B}(H)$ be bounded below and $T^{*n}T^{n+1}(H) \subseteq T(H)$ for all $n \in \mathbb{N}$. Let N be the orthogonal complement of the range of T. Then:

(i) $H = \sum_{n=0}^{\infty} \oplus T^n(N) \oplus \bigcap_{n=1}^{\infty} T^n(H)$.

(ii) *The subspace $\bigcap_{n=1}^{\infty} T^n(H)$ is reducing for T and T restricted to it is an invertible operator.*

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Example

Take $H = H^2(\beta)$ and $T = T_z$ where

$$\beta_n = \begin{cases} \frac{1}{2^{n/2}} & \text{if} \quad n \text{ even}, \\ \frac{1}{2^{(n-1)/2}} & \text{if} \quad n \text{ odd}. \end{cases}$$

Outline of the proof

- Using the lemma,

$$M = \sum_{n=0}^{\infty} T_z^n(N) \oplus \bigcap_{n=1}^{\infty} T_z^n(M),$$

  where $N = M \ominus T_z(M)$.
- Since elements of M are analytic on $\mathbb{D}$, $\bigcap_{n=1}^{\infty} T_z^n(M) = \{0\}$.

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- $dim(N) = 1$.
- Take b a unit vector in N. Then $\{bz^n\}_{n=0}^{\infty}$ is an orthogonal basis for M. Therefore, bH^2 is dense in M.

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- $dim(N) = 1$.
- Take b a unit vector in N. Then $\{bz^n\}_{n=0}^{\infty}$ is an orthogonal basis for M. Therefore, bH^2 is dense in M.
- T_z shifts this orthogonal basis.

Theorem (A. Shields,1974)

$T \in \mathcal{B}(H)$ is an injective weighted shift if and only if T shifts an orthogonal basis of H.

Theorem

Let M be a Hilbert space contained in H^2. Suppose the operator T_z, which denotes multiplication by z, is well defined on M, and satisfies:

(i) *There exists a $\delta > 0$ such that $\delta||f||_M \leq ||T_z f||_M \leq ||f||_M$ for all $f \in M$.*

(ii) *For each $n \in \mathbb{N}$, $T_z^{*n} T_z^{n+1}(M) \subseteq T_z(M)$ (the adjoint of T_z is with respect to the inner product on M).*

Then T_z acts as a weighted shift on M, and there exists a $b \in H^\infty$ such that

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Example

Choose $\{\beta_n\}$ such that $c \leq \beta_{n+1} \leq \beta_n$ for some $c > 0$ and for all n.

Take $\beta_n = (n+3)^{1/(n+3)}$ for $n \geq 0$.

Important consequences

Corollary (Singh and Singh, 1991)

Let M be a Hilbert space contained in H^2 as a vector subspace and such that $T_z(M) \subseteq M$ and let T_z act isometrically on M. Then there exists a $b \in H^\infty$ such that $M = bH^2$, and $||bf||_M = ||f||_2$ for all $f \in H^2$.

The above result generalizes the result of de Branges and therefore of Beurling as well. Hence our result also implies these two classical results.