Hyperrigid generators in C^*-algebras

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The classical theorems of Korovkin impressed several mathematicians since their discovery for the simplicity and the potential. Positive approximation process play a fundamental role in the approximation theory and it appears in a very natural way in several problems dealing with the approximation of continuous functions and qualitative properties such as monotonicity, convexity, shape preservation and so on.

Korovkin Theorem

The classical approximation theorem due to Korovkin in 1953 unified many existing approximation processes

Theorem

If a sequence of positive linear maps $\phi_n : C[0,1] \to C[0,1]$, $n = 1, 2, 3, \ldots,$ *has the property*

$$\lim_{n\to\infty} ||\phi_n(f_k) - f_k|| = 0, \quad k = 0, 1, 2,$$

for the three functions $f_0(x) = 1, f_1(x) = x, f_2(x) = x^2$ *then*

$$\lim_{n\to\infty} ||\phi_n(f) - f|| = 0, \quad \forall \quad f \in C[0,1].$$

Korovkin Set

The set $\{1, x, x^2\}$ is called a *Korovkin set* or *test set*. Korovkin showed that, the set $\{1, x\}$ is not a Korovkin set. Therefore, the set $\{1, x, x^2\}$ is a minimal set to satisfy the above assertion.

Korovkin's theorem generated considerable activity among researchers in approximation theory. The generalizations make essential use of the *Choquet boundary* in one way or another.

Choquet boundary and Saskin's theorem

Definition

Let $S \subset C(X)$ containing the constant function 1, where X is a compact Hausdorff space. The Choquet boundary ∂S of S is defined as $\partial S = \{x \in X : \varepsilon_{x|_S}$ has a unique positive linear extension to $C(X)$, where ε_x denotes the evaluation functional defined by $\varepsilon_x(f) = f(x), f \in C(X)\}$.

Theorem

Let S be a subset of $C(X)$ that separates points of X and contains constant function. Then S is a Korovkin set in $C(X)$ if and only if the Choquet boundary $\partial G = X$. Where $G = linear\ span(S)$

In 2011, Arveson initiated the study of noncommutative approximation theory focusing on the question: How does one determine whether a set of generators of a C^*-algebra is hyperrigid?

Definition

A finite or countably infinite set $\mathscr{G}$ of generators of a C^*-algebra $\mathscr{A}$ is said to be *hyperrigid* if for every faithful representation $\mathscr{A} \subseteq \mathscr{B}(H)$ of $\mathscr{A}$ on a Hilbert space H and every sequence of unital completely positive (UCP) maps $\phi_n : \mathscr{B}(H) \to \mathscr{B}(H)$, $n = 1, 2, \ldots,$

$$\lim_{n\to\infty} ||\phi_n(g) - g|| = 0, \ \forall\, g \in \mathscr{G} \Longrightarrow \lim_{n\to\infty} ||\phi_n(a) - a|| = 0, \ \forall\, a \in \mathscr{A}.$$

Note that, a set $\mathscr{G}$ is hyperrigid if and only if $\mathscr{G} \cup \mathscr{G}^*$ is hyperrigid if and only if the linear span of $\mathscr{G}$ is hyperrigid. If $\mathscr{A}$ is unital, then $\mathscr{G}$ is hyperrigid if and only if $\mathscr{G} \cup \{1\}$ is hyperrigid.

Non-commutative Choquet boundary

In 1969 Arveson introduced the notion of boundary representation.

Definition

Let $\mathscr{S}$ be an operator system in a C^*-algebra $\mathscr{A}$ such that $\mathscr{A} = C^*(\mathscr{S})$. A representation $\pi : \mathscr{A} \to \mathscr{B}(H)$ is said to have unique extension property (UEP) for $\mathscr{S}$, if the only unital completely positive (UCP) map $\phi : \mathscr{A} \to \mathscr{B}(H)$ that satisfies $\phi_{|_{\mathscr{S}}} = \pi_{|_{\mathscr{S}}}$ is $\phi = \pi$ itself.

Definition

Let $\mathscr{S}$ be an operator system in a C^*-algebra $\mathscr{A}$ such that $\mathscr{A} = C^*(\mathscr{S})$. An irreducible representation $\pi : \mathscr{A} \to \mathscr{B}(H)$ of $\mathscr{A}$ is said to be a boundary representation for $\mathscr{S}$ if π has unique extension property (UEP) for $\mathscr{S}$.

Theorem

For every separable operator system $\mathscr{S}$ that generates a C^*-algebra $\mathscr{A} = C^*(\mathscr{S})$, such that $\mathscr{S}$ is hyperrigid if and only if for every nondegenerate representation $\pi : C^*(\mathscr{S}) \to \mathscr{B}(H)$ on a separable Hilbert space, $\pi_{|_{\mathscr{S}}}$ has unique extension property.

Theorem

Let $\mathscr{S}$ be a separable operator system generating a C^*-algebra $\mathscr{A}$ such that $\mathscr{A} = C^*(\mathscr{S})$. If $\mathscr{S}$ is hyperrigid, then every irreducible representation of $\mathscr{A}$ is a boundary representation for $\mathscr{S}$.

Arveson's hyperrigidity conjecture

Conjecture

If every irreducible representation of a C^*-algebra is a boundary representation for a separable operator system then the operator system is hyperrigid.

In 2011, Arveson showed that the hyperrigidity conjecture is true for C^*-algebras with countable spectrum. In 2014, Kleski established the hyperrigidity conjecture for all type-I C^*-algebras with additional assumptions on the co-domain. Davidson and Kennedy proved the conjecture for function systems. Clouatre established the hyperrigidity conjecture with assumption of unperforated pair of subspaces. The hyperrigidity conjecture is still open for general C^*-algebras. Namboodiri, Pramod, Shankar and Vijayarajan approached the hyperrigidity conjecture with weaker notions. They got the partial answers.

The interesting examples of hyperrigid generators are obtained by a direct application. Arveson established the noncommutative strengthening of a classical approximation-theoretic result of Korovkin.

Theorem

Let $X \in \mathscr{B}(H)$ be a self adjoint operator with atleast 3 points in its spectrum and let $\mathscr{A}$ be the C^*-algebra generated by X. Then

(i) $\mathscr{G} = \{X, X^2\}$ *is a hyperrigid generator for* $\mathscr{A}$, *while*

(ii) $\mathscr{G}_0 = \{X\}$ *is not hyperrigid generator for* $\mathscr{A}$.

Theorem

Let $V \in \mathscr{B}(H)$ be an isometry that generates a C^*-algebra $\mathscr{A}$. Then $\mathscr{G} = \{V, VV^*\}$ is hyperrigid generator for $\mathscr{A}$.

Essential use of noncommutative Choquet boundary.

Theorem

Let $V \in \mathscr{B}(H)$ be an irreducible compact operator with cartesian decomposition $V = A + iB$, where A is a finite rank positive operator and B is essential with $KerB = \{0\}$. Then

(i) *$\mathscr{G} = \{V, V^2\}$ is hyperrigid generator for C^*-algebra $\mathscr{K}(H)$ of compact operators. In particular every sequence of unital completely positive maps $\phi_n : \mathscr{B}(H) \to \mathscr{B}(H)$ for which*

$$\lim_{n\to\infty} ||\phi_n(V) - V|| = \lim_{n\to\infty} ||\phi_n(V^2) - V^2|| = 0,$$

one has

$$\lim_{n\to\infty} ||\phi_n(K) - K|| = 0$$

for every compact operator $K \in \mathscr{B}(H)$.

(ii) *The smaller generating set $\mathscr{G}_0 = \{V\}$ of $\mathscr{K}(H)$ is not hyperrigid.*

Let $S = (S_1, ..., S_d)$ denote the compression of the d-shift to the complement of a homogeneous ideal I of $\mathbb{C}[z_1, ..., z_d]$. Following the remark above, in 2016, Kennedy and Shalit proved that, if homogeneous ideals are sufficiently non-trivial then S is essentially normal if and only if it is hyperrigid as the generating set of a C^*-algebra.

Essential Unitary and Hyperrigidity

Let $\mathscr{B}(H)$ be the algebra of bounded linear operators on a separable complex Hilbert space H and $\mathscr{K}(H)$ ideal of compact operators on H. Let $\pi : \mathscr{B}(H) \to \mathscr{B}(H)/\mathscr{K}(H)$ be the natural surjection onto the Calkin algebra $\mathscr{B}(H)/\mathscr{K}(H)$. The operator $T \in \mathscr{B}(H)$ is called essentially normal if $\pi(T)$ is normal in the Clakin algebra, or equivalently, $T^*T - TT^*$ is compact. The operator $S \in \mathscr{B}(H)$ is called essentially unitary if $\pi(S)$ is unitary in the Clakin algebra, or equivalently, $I - S^*S$ and $I - SS^*$ are compact.

Here, we will have the following assumptions to proceed. Let S be a irreducible and essential unitary but not unitary operator in $\mathscr{B}(H)$ and let $\mathscr{G} = \{S, SS^*\}$. Let $\mathscr{S}$ be a operator system generated by $\mathscr{G}$. Let $\mathscr{T} = C^*(\mathscr{G})$ be the unital C^*-algebra generated by $\mathscr{G}$. The unital C^*-algebra $\mathscr{T}$ contains the compact operators $\mathscr{K}(H)$.

Definition

A representation $\rho : \mathscr{T} \to \mathscr{B}(H)$ is said to be singular representation if it annihilates the compact operators $\mathscr{K}(H)$.

Lemma

Let $\rho : \mathscr{T} \to \mathscr{B}(H)$ be a representation, and let $\pi : \mathscr{T} \to \mathscr{B}(K)$ be a representation such that $\pi|_{\mathscr{S}}$ is a dilation of $\rho|_{\mathscr{S}}$. Then the subspace H is coinvariant for $\pi(\mathscr{S})$.

Proof:

With respect to the decomposition $K = H \oplus H^{\perp}$. By assumption we have

$$\pi(S) = \begin{pmatrix} \rho(S) & X \\ Y & Z \end{pmatrix}$$

Note that $X = P_H \pi(S)|_{H^{\perp}}$. We must prove that $X = 0$. By assumption,

$$\pi(SS^*) = \begin{pmatrix} \rho(SS^*) & X_0 \\ Y_0 & Z_0 \end{pmatrix}$$

$$\pi(S)\pi(S)^* = \begin{pmatrix} \rho(S) & X \\ Y & Z \end{pmatrix} \begin{pmatrix} \rho(S)^* & Y^* \\ X^* & Z^* \end{pmatrix}.$$

We get,

$$\rho(SS^*) = \rho(S)\rho(S)^* + XX^*$$

Therefore, $XX^* = 0$, and hence $X = 0$.

Proposition

Suppose that S is irreducible and essential unitary and $\mathscr{G} = \{S, SS^*\}$. Let $\mathscr{S}$ be a operator system generated by $\mathscr{G}$ and $\mathscr{T} = C^*(\mathscr{G})$. Let $\rho : \mathscr{T} \to \mathscr{B}(H)$ be a singular representation. Then the restriction $\rho|_{\mathscr{S}}$ has unique extension property.

Proof:

We will use the fact that a UCP map ϕ' has the unique extension property if and only if ϕ' is *maximal*, meaning that every UCP map that dilates ϕ' contains as a direct summand. Let K be a Hilbert space properly containing H. Let $\pi : \mathscr{T} \to \mathscr{B}(K)$ be a representation such that the restriction $\pi|_{\mathscr{S}}$ is a dilation of $\rho|_{\mathscr{S}}$. To show that the restriction $\rho|_{\mathscr{S}}$ has unique extension property, it is enough to show that the dilation π is trivial, that is, $\pi|_{\mathscr{S}} = \rho|_{\mathscr{S}} \oplus \phi$ for some UCP map ϕ.

Using the Lemma, we can decompose $K = H \oplus H^{\perp}$ and write

$$\pi(S) = \begin{pmatrix} \rho(S) & 0 \\ Y & Z \end{pmatrix}.$$

Since ρ is singular, $\rho(S)$ is unitary, so it cannot be dilated to a compression. Therefore the dilation π must be trivial.

Proposition

Suppose that S is irreducible and essential unitary and $\mathscr{G} = \{S, SS^*\}$. Let $\mathscr{S}$ be a operator system generated by $\mathscr{G}$ and $\mathscr{T} = C^*(\mathscr{G})$. Then the identity representation of $\mathscr{T}$ is a boundary representation for $\mathscr{S}$.

Proof:

Since S is irreducible and essential unitary. The unital C^*-algebra generated by $\mathscr{G}$ contains the compact operators, that is, $\mathscr{K}(H) \subseteq \mathscr{T} = C^*(\mathscr{G})$. The operator system $\mathscr{S} \subset \mathscr{T}$ is irreducible and contains the identity operator. By our assumption, $0 \neq \mathscr{K} = I - SS^* \in \mathscr{S}$ is a compact operator, we have $||\mathscr{K} - \mathscr{K}|| < ||\mathscr{K}||$. Therefore, the quotient map $q : \mathscr{B}(H) \to \mathscr{B}(H)/\mathscr{K}(H)$ is not completely isometric on $\mathscr{S}$. Hence by boundary theorem of Arveson, identity representation of $\mathscr{T}$ is a boundary representation for $\mathscr{S}$.

Theorem

Let S be an irreducible and essential unitary and $\mathscr{G} = \{S, SS^*\}$. Let $\mathscr{T} = C^*(\mathscr{G})$ be the unital C^*-algebra generated by $\mathscr{G}$. Then $\mathscr{G}$ is a hyperrigid generator for $\mathscr{T}$.

Proof:

Let $\mathscr{S}$ be the operator system generated by $\mathscr{G}$. Note that $\mathscr{G}$ is hyperrigid if and only if $\mathscr{S}$ is hyperrigid. It suffices to show that for every nondegenerate representation ρ of $\mathscr{T}$, $\rho|_{\mathscr{S}}$ has the unique extension property.

The Proposition implies that every singular nondegenerate representation π of $\mathscr{T}$, $\pi|_{\mathscr{S}}$ has the unique extension property. By Proposition, the restriction of the identity representation of $\mathscr{T}$ to $\mathscr{S}$ has the unique extension property. Since every nondegenerate representation of $\mathscr{T}$ splits as the direct sum of a multiple of the identity representation and a singular nondegenerate representation and by the unique extension property passes to direct sums. Hence every nondegenerate representation of $\mathscr{T}$ restricted to $\mathscr{S}$ has the unique extension property.

Here, We discuss the hyperrigid generators for the C^*-algebras generated by a single operator.

Theorem

Let T be an operator in $\mathscr{B}(H)$ that generate a unital C^*-algebra $\mathscr{A}$ and let $\mathscr{G} = \{T, T^*T, TT^*\}$. Then $\mathscr{G}$ is hyperrigid generators for unital C^*-algebra $\mathscr{A}$.

Proof:

Let $\mathscr{S}$ be the operator system generated by $\mathscr{G}$. It suffices to show that for every nondegenerate representation π of $\mathscr{A}$, $\pi|_{\mathscr{S}}$ has the unique extension property.

Let $\pi : \mathscr{A} \to \mathscr{B}(H)$ be a nondegenerate representation. Let $\phi : \mathscr{A} \to \mathscr{B}(H)$ be a UCP map satisfying $\phi(T) = \pi(T), \phi(T^*T) = \pi(T^*T)$ and $\phi(TT^*) = \pi(TT^*)$. We have to show that $\phi = \pi$ on $\mathscr{A}$.

Using Stinespring theorem, we can express ϕ in the form

$$\phi(S) = V^*\sigma(S)V, \ \ \forall \ \ S \in \mathscr{A}.$$

Where σ is a representation of A on a Hilbert space K,
$V : H \to K$ is an isometry, and which is minimal in the sense that $\overline{\sigma(A)VH} = K$.
We first claim that $\sigma(T)V = V\pi(T)$, We have

$$V^*\sigma(T)^*VV^*\sigma(T)V = \phi(T)^*\phi(T) = \pi(T)^*\pi(T) = \pi(T^*T)$$

Hence,

$$\begin{aligned} & V^*\sigma(T)^*(1 - VV^*)\sigma(T)V \\ = \ & V^*\sigma(T)^*\sigma(T)V - V^*\sigma(T)^*VV^*\sigma(T)V \\ = \ & V^*\sigma(T^*T)V - \pi(T)^*\pi(T) \\ = \ & \pi(T^*T) - \pi(T^*T) = 0. \end{aligned}$$

$\sigma(T)$ leaves VH invariant.

Therefore $\sigma(T)V = VV^*\sigma(T)V = V\phi(T) = V\pi(T)$.

$$\begin{aligned} & VV^*\sigma(T)(1_K - VV^*)\sigma(T)^*VV^* \\ = \;& VV^*\sigma(T)\sigma(T)^*VV^* \\ & -VV^*\sigma(T)VV^*\sigma(T)^*VV^* \\ = \;& VV^*\sigma(TT^*)VV^* - V\pi(T)\pi(T)^*V^* \\ = \;& V\pi(TT^*)V^* - V\pi(TT^*)V^* = 0. \end{aligned}$$

Hence $(1_K - VV^*)\sigma(T)^*VV^* = 0$, we conclude that VH is invariant under both $\sigma(T)$ and $\sigma(T)^*$. Since $\mathscr{A}$ is generated by T it follows that $\sigma(\mathscr{A})VH \subseteq VH$. By minimality we must have $VH = K$, which implies that V is unitary and therefore $\phi(S) = V^{-1}\sigma(S)V$ is a representation. Since ϕ agrees with π on a generating set. Therefore $\phi = \pi$ on $\mathscr{A}$.

Corollary

Let T be a normal operator in $\mathscr{B}(H)$ that generate a unital C^*-algebra $\mathscr{A}$ and let $\mathscr{G} = \{T, TT^*\}$. Then $\mathscr{G}$ is hyperrigid generator for unital C^*-algebra $\mathscr{A}$.

Corollary

Let T be an unitary operator in $\mathscr{B}(H)$ that generate a C^*-algebra $\mathscr{A}$ and let $\mathscr{G} = \{T\}$. Then $\mathscr{G}$ is hyperrigid generator for C^*-algebra $\mathscr{A}$.

Proposition

Let $V \in \mathscr{B}(H)$ be an isometry (not unitary) that generates a C^*-algebra $\mathscr{A}$. Then

(i) $\mathscr{G} = \{V, VV^*\}$ *is hyperrigid generator for* $\mathscr{A}$.

(ii) *The smaller generating set* $\mathscr{G}_0 = \{V\}$ *is not hyperrigid.*

Proof:

Let $\mathscr{S}$ be the operator system generated by $\mathscr{G}_0$. Let Id denote the identity representation of a C^*-algebra $\mathscr{A}$. Let $V^*Id(\cdot)V$ be a completely positive map on the C^*-algebra $\mathscr{A}$. We have $V^*IdV|_{\mathscr{S}} = Id|_{\mathscr{S}}$, but $V^*Id(VV^*)V = I \neq VV^* = Id(VV^*)$. This implies that Id representation restricted to $\mathscr{S}$ has two UCP map extensions V^*IdV and Id. Therefore the nondegenerate representation $Id|_{\mathscr{S}}$ does not have unique extension property. $\mathscr{S}$ is not hyperrigid operator system in a C^*-algebra $\mathscr{A}$. This will imply that $\mathscr{G}_0$ is not hyperrigid in $\mathscr{A}$.

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Thank You