Applications of de Branges-Rovnyak decomposition to Graph Theory [1]

Michio Seto (National Defense Academy)

Joint work with

Sho Suda (Aichi University of Education)

and

Tetsuji Taniguchi (Hiroshima Institute of Technology)

[1] Supported by JSPS KAKENHI Grant Numbers JP15K04926 and JP16K05190.

Reminiscences of Bieberbach conjecture

$$\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$$

$$\mathcal{S} := \{f \in \mathrm{Hol}(\mathbb{D}) : f \text{ is injective and } f(z) = z + \sum_{n=2}^{\infty} c_n z^n\}$$

Bieberbach conjecture

- If $f(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{S}$, then $|c_n| \leq n$?

- In 1984, de Branges gave the solution with Hilbert space operator theory.

- However, since his original proof was very complicated, his operator theory method has been forgotten.

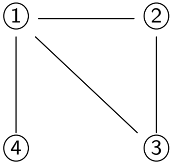
In this talk

- We deal with increasing sequences of graphs from the viewpoint of Hilbert space operator theory.
- As results, two different types of inequality are given.
- Our scheme gives a toy model of de Branges' solution to the Bieberbach conjecture (in fact, this is my motivation).

Preliminaries from Graph theory

Graph

We deal with simple graphs (no loops, no multi-edges and no direction).



$$V = \{1, 2, 3, 4\}, \quad E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}\}, \quad G = (V, E)$$

Laplace matrix

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \quad (\rightarrow \text{spectral graph theory}).$$

Inequality 1

Setting

- V: a finite set of vertices (fixed),
- $G_j = (V, E_j)$: connected and simple graphs

$$\text{s.t.} \quad G_1 \subset \cdots \subset G_n \quad (\text{i.e.} \quad E_1 \subset \cdots \subset E_n).$$

- $\gamma(G)$: the number of connected components of G.

Inequality 1

Setting

- V: a finite set of vertices (fixed),
- $G_j = (V, E_j)$: connected and simple graphs

$$\text{s.t.} \quad G_1 \subset \cdots \subset G_n \quad (\text{i.e.} \quad E_1 \subset \cdots \subset E_n).$$

- $\gamma(G)$: the number of connected components of G.

Inequality for γ

$$\sum_{j=1}^{n-1} \gamma(G_{j+1} - G_j) \leq \gamma(G_n - G_1) + (n-2)|V|$$

$$(G_{j+1} - G_j := (V, E_{j+1} \setminus E_j)).$$

- We have an operator theory proof of this inequality.

Inequality 2

Setting

- V: a finite set (fixed),
- $G_j = (V, E_j)$ $(j = 0, 1)$: connected and simple graphs s.t.

$$G_0 \subset G_1 \quad \text{(i.e. } G_0 \text{ is a subgraph of } G_1\text{)}$$

- L_j: Laplace matrix of G_j,
- $K_j = (P + L_j)^{-1}$ (where $P := \operatorname{proj} \ker L_j$ in $\ell^2(V)$)

Inequality 2

Setting

- V: a finite set (fixed),
- $G_j = (V, E_j)$ $(j = 0, 1)$: connected and simple graphs s.t.

$$G_0 \subset G_1 \quad \text{(i.e. } G_0 \text{ is a subgraph of } G_1\text{)}$$

- L_j: Laplace matrix of G_j,
- $K_j = (P + L_j)^{-1}$ (where $P := \operatorname{proj} \ker L_j$ in $\ell^2(V)$)

Trivial observation

- $G_0 \subset G_1 \;\Rightarrow\; L_0 \leq L_1 \;\Leftrightarrow\; P + L_0 \leq P + L_1 \;\Leftrightarrow\; K_0 \geq K_1.$
- Many graph theorists are interested in spectral property of L. We shall improve $K_0 \geq K_1$ in the next page.

Inequality 2

Theorem (S-Suda)

If $G_0 \subset G_1$ (simple, connected and having the same vertex set), then $\forall c \in \ell^2(V)$

$$0 \leq \langle L_0(K_0 - K_1)\widetilde{c}, (K_0 - K_1)\widetilde{c}\rangle_{\ell^2(V)} \leq \langle (K_0 - K_1)c, c\rangle_{\ell^2(V)},$$

where

$$\widetilde{c} := \frac{1}{|\mathcal{G}|}\sum_{g \in \mathcal{G}} c \circ g,$$

the averaged vector of c with respect to

$$\mathcal{G} = \mathrm{Aut}(G_0) \cap \mathrm{Aut}(G_1).$$

Remarks

- These two inequalities are derived from de Branges-Rovnyak theory.
- Our scheme gives general method for finding inequalities (but it is rather complicated).
- There is a proof of Inequality 1 with graph theory (Ozeki).
- We have a simple proof of Inequality 2 without de Branges-Rovnyak theory.

Our idea

Hilbert space $\mathcal{H}_G$

- For functions u and v on V (in fact, u and v are vectors),

$$\langle u, v\rangle_{\mathcal{H}_G} := \langle (I_{\ell^2(V)} + L_G)u, v\rangle_{\ell^2(V)} \quad (L_G\text{: Laplacian of } G).$$

- $\mathcal{H}_G$: the Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_G}$

Our idea

Hilbert space $\mathcal{H}_G$

- For functions u and v on V (in fact, u and v are vectors),

$$\langle u, v\rangle_{\mathcal{H}_G} := \langle (I_{\ell^2(V)} + L_G)u, v\rangle_{\ell^2(V)} \quad (L_G\text{: Laplacian of } G).$$

- $\mathcal{H}_G$: the Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_G}$

Translation from G to $\mathcal{H}$

$$G_1 \subset G_2 \subset \cdots \subset G_{n-1} \subset G_n$$

$$\downarrow$$

$$\mathcal{H}_{G_1} \hookleftarrow \mathcal{H}_{G_2} \hookleftarrow \cdots \hookleftarrow \mathcal{H}_{G_{n-1}} \hookleftarrow \mathcal{H}_{G_n}$$

Can this sequence be telescoped?

(Note: $\mathcal{H}_{G_{j+1}}$ is not a Hilbert subspace of $\mathcal{H}_{G_j}$)

Our idea

Hilbert space $\mathcal{H}_G$

- For functions u and v on V (in fact, u and v are vectors),

$$\langle u, v\rangle_{\mathcal{H}_G} := \langle (I_{\ell^2(V)} + L_G)u, v\rangle_{\ell^2(V)} \quad (L_G\text{: Laplacian of } G).$$

- $\mathcal{H}_G$: the Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_G}$

Translation from G to $\mathcal{H}$

$$\begin{gathered} G_1 \subset G_2 \subset \cdots \subset G_{n-1} \subset G_n \\ \downarrow \\ \mathcal{H}_{G_1} \hookleftarrow \mathcal{H}_{G_2} \hookleftarrow \cdots \hookleftarrow \mathcal{H}_{G_{n-1}} \hookleftarrow \mathcal{H}_{G_n} \end{gathered}$$

Can this sequence be telescoped?

(Note: $\mathcal{H}_{G_{j+1}}$ is not a Hilbert subspace of $\mathcal{H}_{G_j}$)

Our answer

Use de Branges-Rovnyak theory.

A review of de Branges-Rovnyak theory

Pull-back construction

- $\mathcal{H}, \mathcal{K}$: Hilbert spaces
- $T : \mathcal{H} \to \mathcal{K}$ any bounded linear operator,
- $\langle Tx, Ty \rangle_T := \langle Px, Py \rangle_{\mathcal{H}}$ $(P := P_{(\ker T)^\perp})$,
- $\mathcal{M}(T) := (T\mathcal{H}, \langle \cdot, \cdot \rangle_T)$ is a Hilbert space

$$\because \quad T\mathcal{H} \cong \mathcal{H}/\ker T \cong (\ker T)^\perp.$$

A review of de Branges-Rovnyak theory

Fundamental theorem

If $T : \mathcal{H} \to \mathcal{K}$ and $\|T\| \leq 1$, then

1. $\mathcal{K} = \mathcal{M}(T) + \mathcal{H}(T) \quad (\mathcal{H}(T) := \mathcal{M}(\sqrt{I_{\mathcal{K}} - TT^*}))$,
2. $\|z\|^2_{\mathcal{K}} \leq \|x\|^2_{\mathcal{M}(T)} + \|y\|^2_{\mathcal{H}(T)}$ if $z = x + y \in \mathcal{M}(T) + \mathcal{H}(T)$
3. $\forall z \in \mathcal{K} \quad \exists! x_z \in \mathcal{M}(T) \quad \exists! y_z \in \mathcal{H}(T)$

   s.t. $\quad z = x + y \quad$ and $\quad \|z\|^2_{\mathcal{K}} = \|x_z\|^2_{\mathcal{M}(T)} + \|y_z\|^2_{\mathcal{H}(T)}$

- See Ando's lecture notes or Sarason's red book for details.

How to use de Branges-Rovnyak theory in graph theory

Increasing family of graphs

$G_1 \subset G_2 \subset \cdots \subset G_{n-1} \subset G_n$

Embedding of Hilbert spaces

$\mathcal{H}_{G_1} \overset{T_{1,2}}{\longleftarrow} \mathcal{H}_{G_2} \overset{T_{2,3}}{\longleftarrow} \cdots \overset{T_{n-2,n-1}}{\longleftarrow} \mathcal{H}_{G_{n-1}} \overset{T_{n-1,n}}{\longleftarrow} \mathcal{H}_{G_n}$

Telescoping

$T_1 := I_{\mathcal{H}_{G_1}}, T_{j+1} := T_j T_{j,j+1},$

$\mathcal{H}(T_n) = \sum_{j=1}^{n-1} \mathcal{M}(\sqrt{T_j T_j^* - T_{j+1} T_{j+1}^*}).$

Trivial estimate

$\dim \mathcal{H}(T_n) \leq \sum_{j=1}^{n-1} \dim \mathcal{M}(\sqrt{T_j T_j^* - T_{j+1} T_{j+1}^*})$

($\Rightarrow$ Inequality 1).

How to use de Branges-Rovnyak theory in graph theory

Time evolution of graphs

$$G_0 \subset G_1 \quad \to \quad G_0 \subset G_r \subset G_t \subset G_1 \quad (0 \leq r \leq t \leq 1).$$

Continuous chain of Hilbert spaces

$$\mathcal{H}_{G_0} \hookleftarrow \mathcal{H}_{G_r} \overset{T_{rt}}{\hookleftarrow} \mathcal{H}_{G_t} \hookleftarrow \mathcal{H}_{G_1} \quad (0 \leq r \leq t \leq 1).$$

Quasi-orthogonal integrals
(for details, Vasyunin-Nikolskii, Leningrad Math. J. (1991).)

$$\mathcal{H}(T_{rt}) = \int_r^t \mathcal{M}(T_{rs}\Delta(s))\, ds \quad (0 \leq r \leq t \leq 1).$$

$$\| \int_r^t T_{rs}\Delta(s)f(s)\, ds\|^2_{\mathcal{H}(T_{rt})} \leq \int_r^t \|\Delta(s)f(s)\|^2_{\mathcal{M}(\Delta(s))}\, ds$$

($\Rightarrow$ Inequality 2).

Summary 1

The following inequalities are derived from de Branges-Rovnyak theory (discrete and continuous cases):

- If $G_1 \subset G_2 \subset \cdots \subset G_{n-1} \subset G_n$, then

$$\sum_{j=1}^{n-1} \gamma(G_{j+1} - G_j) \leq \gamma(G_n - G_1) + (n-2)|V|.$$

- If $G_0 \subset G_1$, then

$$0 \leq \langle L_0(K_0 - K_1)\widetilde{c}, (K_0 - K_1)\widetilde{c}\rangle_{\ell^2(V)} \leq \langle (K_0 - K_1)c, c\rangle_{\ell^2(V)}$$

$$(c \in \ell^2(V)).$$

Summary 2

Our scheme

$\boxed{\text{increasing sequences of non-negative matrices}}$

$\downarrow$ input

$\boxed{\text{de Branges-Rovnyak theory (quasi-orthgonal integrals)}}$

$\downarrow$ output

$\boxed{\text{inequalities}}$

This device is similar to that many identities are implied from formulas in Fourier analysis.