K-contractions

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Joint work with Jörg Eschmeier (Saarland University)

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Let $\mathcal{H}$ be a complex Hilbert space.

Lemma

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2. *$1/K(T, T^*) \geq 0$, where*

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}, \ (z, w) \mapsto \frac{1}{1 - z\overline{w}} = \frac{1}{1 - \langle z, w \rangle}$$

is the reproducing kernel of the Hardy space on the unit disc $H^2(\mathbb{D})$.

Example

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Lemma

1. *If $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ are contractions, then $T \oplus S$ is a contraction.*
2. *If $T \in B(\mathcal{H})$ is a contraction and $\mathcal{M} \subset \mathcal{H}$ is a subspace, then $T|_{\mathcal{M}}$ is a contraction.*

Definition

We say that a contraction $T \in B(\mathcal{H})$ belongs to the class $C_{\cdot 0}$ or is *pure* if

$$T_\infty = \tau_{\mathrm{SOT}}\text{-}\lim_{N \to \infty} T^N T^{*N} = 0.$$

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Example

The shift operator $M_z \in B(H^2(\mathbb{D}))$ belongs to $C_{\cdot 0}$.

Theorem

Let $T \in B(\mathcal{H})$ be a contraction. Then

$$\pi_T : \mathcal{H} \to H^2(\mathbb{D}, \mathcal{D}_T), \; h \mapsto \sum_{n=0}^{\infty} (D_T T^{*n} h) z^n,$$

where $D_T = (1 - TT^*)^{1/2} = (1/K(T, T^*))^{1/2}$ and $\mathcal{D}_T = \overline{D_T \mathcal{H}}$, is a well-defined bounded linear operator.

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$$\|\pi_T h\|^2 = \|h\|^2 - \langle T_\infty h, h \rangle$$

for all $h \in \mathcal{H}$, *and*

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Corollary

A contraction $T \in B(\mathcal{H})$ is in $C_{\cdot 0}$ if and only if π_T is an isometry.

The $C_{\cdot 0}$ case

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Let $T \in B(\mathcal{H})$ be an operator. The following statements are equivalent:

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Let $T \in B(\mathcal{H})$ be an operator. The following statements are equivalent:

1. *T is a contraction which belongs to $C_{\cdot 0}$,*
2. *there exist a Hilbert space $\mathcal{D}$, and an isometry $\pi \colon \mathcal{H} \to H^2(\mathbb{D}, \mathcal{D})$ such that*

$$\pi T^* = (M_z^{\mathcal{D}})^* \pi.$$

A Beurling-type theorem

Remark

If $T \in B(\mathcal{H})$ is a $C_{\cdot 0}$-contraction and $\mathcal{S} \in \mathrm{Lat}(T)$, then $T|_{\mathcal{S}}$ is also $C_{\cdot 0}$-contraction.

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1. $\mathcal{S} \in \mathrm{Lat}(M_z^{\mathcal{E}})$,
2. *there exist a Hilbert space $\mathcal{D}$, and a bounded analytic function $\theta \colon \mathbb{D} \to B(\mathcal{D}, \mathcal{E})$ such that*

$$M_\theta \colon H^2(\mathbb{D}, \mathcal{D}) \to H^2(\mathbb{D}, \mathcal{E}), \ f \mapsto \theta f$$

is a partial isometry with $\mathrm{Im}(M_\theta) = \mathcal{S}$.

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Question

For which reproducing kernels K does an analogue theorem hold? What happens if we look at commuting tuples $T = (T_1, \ldots, T_d) \in B(\mathcal{H})^d$?

Unitarily invariant spaces on $\mathbb{B}_d$

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers with $a_0 = 1$ and such that

$$k(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (z \in \mathbb{D})$$

defines a holomorphic function with radius of convergence at least 1 and no zeros in the unit disc $\mathbb{D}$.

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$$K\colon \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C},\ (z, w) \mapsto \sum_{n=0}^{\infty} a_n \langle z, w\rangle^n$$

defines a semianalytic positive definite function and hence, there exists a reproducing kernel Hilbert space $H_K \subset \mathcal{O}(\mathbb{B}_d)$ with kernel K.

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defines a semianalytic positive definite function and hence, there exists a reproducing kernel Hilbert space $H_K \subset \mathcal{O}(\mathbb{B}_d)$ with kernel K. The space H_K is a so called *unitarily invariant space on* $\mathbb{B}_d$.

Furthermore, we suppose that $\sup_{n\in\mathbb{N}} a_n/a_{n+1} < \infty$ such that the K-shift $M_z = (M_{z_1}, \ldots, M_{z_d}) \in B(H_K)^d$ is well-defined.

Furthermore, we suppose that $\sup_{n\in\mathbb{N}} a_n/a_{n+1} < \infty$ such that the K-shift $M_z = (M_{z_1}, \ldots, M_{z_d}) \in B(H_K)^d$ is well-defined.
Since k has no zeros in $\mathbb{D}$, the function

$$\frac{1}{k}\colon \mathbb{D} \to \mathbb{C},\ z \mapsto \frac{1}{k(z)}$$

is again holomorphic and hence admits a Taylor expansion

$$\frac{1}{k}(z) = \sum_{n=0}^{\infty} c_n z^n \qquad (z \in \mathbb{D})$$

with a suitable sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$.

Example

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2. If $\nu > 0$ and $a_n = a_n^{(\nu)} = (-1)^n \binom{-\nu}{n}$ for all $n \in \mathbb{N}$, then

$$K(z,w) = K^{(\nu)}(z,w) = \frac{1}{(1 - \langle z, w \rangle)^\nu} \qquad (z, w \in \mathbb{B}_d),$$

i.e., $H_{K^{(\nu)}}$ is a *weighted Bergman space*.

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   i.e., $H_{K^{(\nu)}}$ is a *weighted Bergman space*.
3. The space H_K is a *complete Nevanlinna-Pick space* if and only if

$$c_n \leq 0$$

   for all $n \geq 1$.

Definition

Let $T = (T_1, \ldots, T_d) \in B(\mathcal{H})^d$ be a commuting tuple. Define

$$\left(\frac{1}{K}\right)_N (T, T^*) = \sum_{n=0}^{N} c_n \sigma_T^k(1)$$

for all $N \in \mathbb{N}$, where

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Furthermore, we write

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if the latter exists.

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3. Let $m \in \mathbb{N}^*$. We call a commuting tuple $T \in B(\mathcal{H})^d$ an *m-hypercontraction* if T is a $K^{(\ell)}$-contraction for all $\ell = 1, \ldots, m$.

Proposition (Chen, 2012)

If there exists a natural number $p \in \mathbb{N}$ such that

$$c_n \geq 0 \text{ for all } n \geq p \quad \text{or} \quad c_n \leq 0 \text{ for all } n \geq p$$

holds, then

$$\frac{1}{K}(M_z, M_z^*) = P_{\mathbb{C}} \quad \text{and} \quad \sum_{n=0}^{\infty} |c_n| < \infty.$$

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For the rest of this section, we suppose that the condition in the last proposition holds.

Definition

Let $T \in B(\mathcal{H})^d$ be a K-contraction. We define

$$\Sigma_N(T) = 1 - \sum_{n=0}^{N} a_n \sigma_T^n \left(\frac{1}{K}(T, T^*) \right)$$

for $N \in \mathbb{N}$ and write

$$\Sigma(T) = \tau_{\mathrm{SOT}}\text{-}\lim_{N \to \infty} \Sigma_N(T)$$

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Remark

If $K = K^{(1)}$ and $T \in B(\mathcal{H})^d$ is a $K^{(1)}$-contraction, then

$$\Sigma_N(T) = \sigma_T^{N+1}(1)$$

for all $N \in \mathbb{N}$, and hence,

$$\Sigma(T) = T_\infty \geq 0.$$

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where $D_T = (1/K(T, T^*))^{\frac{1}{2}}$ and $\mathcal{D}_T = \overline{D_T \mathcal{H}}$, is a well-defined bounded linear operator. Furthermore, we have

$$\|\pi_T h\|^2 = \|h\|^2 - \langle \Sigma(T)h, h \rangle$$

for all $h \in \mathcal{H}$ and

$$\pi_T T_i^* = (M_{z_i}^{\mathcal{D}_T})^* \pi_T$$

for all $i = 1, \ldots, d$.

The pure case

Theorem (Eschmeier, S.)

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Theorem (Eschmeier, S.)

Let $\mathcal{E}$ be a Hilbert space. For $\mathcal{S} \subset H_K(\mathcal{E})$, the following statements are equivalent:

1. $\mathcal{S} \in \mathrm{Lat}\left(M_z^{\mathcal{E}}\right)$ *and* $M_z^{\mathcal{E}}|_{\mathcal{S}}$ *is K-pure,*

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is a partial isometry with $\operatorname{Im}(M_\theta) = \mathcal{S}$.

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2. Every spherical unitary is a strong K-contraction.

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If we assume that H_K is regular, i.e., $\lim_{n\to\infty} a_n / a_{n+1} = 1$, then the above are also equivalent to

- *there is a unital completely contractive linear map $\rho \colon \operatorname{span}\left\{\mathrm{id}_{H_K}, M_{z_i}, M_{z_i} M_{z_i}^* \;;\; i = 1, \ldots, d\right\} \to B(\mathcal{H})$ with*

$$\rho(M_{z_i}) = T_i, \; \rho(M_{z_i} M_{z_i}^*) = T_i T_i^* \qquad (i = 1, \ldots, d).$$

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Radial K-hypercontractions

Definition

We call a commuting tuple $T \in B(\mathcal{H})^d$ with $\sigma(T) \subset \overline{\mathbb{B}}_d$ a *radial K-hypercontraction* if, for all $0 < r < 1$, $rT \in B(\mathcal{H})^d$ is a K-contraction.

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Example

Spherical unitaries are radial K-hypercontractions.

Remark

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$$k_r : \mathbb{D} \to \mathbb{C}, \; z \mapsto k(rz) \qquad (r \in [0,1]).$$

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For $r, s \in [0,1]$, the function k_s / k_r has a Taylor expansion on $\mathbb{D}$

$$\sum_{n=0}^{\infty} a_n(s,r) z^n.$$

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Note that

$$a_n(1,0) = a_n \quad \text{and} \quad a_n(0,1) = c_n \qquad (n \in \mathbb{N}).$$

Remark (Guo-Hu-Xu, 2004)

If H_K is regular, then $\sigma(M_z) \subset \overline{\mathbb{B}}_d$.

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Proposition (Olofsson, 2015)

The K-shift $M_z \in B(H_K)^d$ is a radial K-hypercontraction if and only if

$$a_n(1, r) \geq 0$$

for all $n \in \mathbb{N}$ and $0 < r < 1$.

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Remark (Olofsson, 2015)

If H_K is a complete Nevanlinna-Pick space or a generalized Bergman space, then M_z is a radial K-hypercontraction.

Proposition (Olofsson, 2015)

Suppose that M_z is a radial K-hypercontraction. Let $T \in B(\mathcal{H})^d$ be a radial K-hypercontraction. Then

$$\frac{1}{K}_{\mathrm{rad}}(T, T^*) = \tau_{\mathrm{SOT}}\text{-}\lim_{r \to 1} \frac{1}{K}(rT, rT^*)$$

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Corollary

If M_z is a radial K-hypercontraction, then

$$\frac{1}{K}_{\mathrm{rad}}(M_z, M_z^*) = P_{\mathbb{C}}.$$

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1. *T is a radial K-hypercontraction,*
2. *there exist Hilbert spaces $\mathcal{D}, \mathcal{K}$, a spherical unitary $U \in B(\mathcal{K})^d$, and an isometry $\Pi \colon \mathcal{H} \to H_K(\mathcal{D}) \oplus \mathcal{K}$ such that*

$$\Pi T_i^* = (M_{z_i}^{\mathcal{D}} \oplus U_i)^* \Pi \qquad (i = 1, \ldots, d),$$

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$$\Pi T_i^* = (M_{z_i}^{\mathcal{D}} \oplus U_i)^* \Pi \qquad (i = 1, \ldots, d),$$
3. *there is a unital completely contractive linear map $\rho \colon S \to B(\mathcal{H})$ on the operator space $S = \operatorname{span}\left\{\mathrm{id}_{H_K}, M_{z_i}, M_{z_i} M_{z_i}^* \; ; \; i = 1, \ldots, d\right\}$ with*
$$\rho(M_{z_i}) = T_i, \;\; \rho(M_{z_i} M_{z_i}^*) = T_i T_i^* \qquad (i = 1, \ldots, d).$$