Classification of crossed products of irrational rotation algebras by cyclic subgroups of $SL_2(\mathbb{Z})$

Sayan Chakraborty (ISI Kolkata)

December 18, 2018

OTOA 2018, Bangalore

A little bit background...

In noncommutative geometry one studies "spaces" in terms of algebras.

A little bit background...

In noncommutative geometry one studies "spaces" in terms of algebras.
Sources of C*-algebras:

- topological spaces
- groups, groupoids,...
- group actions on other C*-algebras

A little bit background...

In noncommutative geometry one studies "spaces" in terms of algebras.
Sources of C*-algebras:

- topological spaces
- groups, groupoids,...
- group actions on other C*-algebras

One basic example (noncommutative torus)

The universal C*-algebra generated by two unitaries U_1 and U_2 with the relation

$$U_2 U_1 = e^{2\pi i\theta} U_1 U_2, \quad \theta \in \mathbb{R} \setminus \mathbb{Q}.$$

We denote the algebra by A_θ.

What is classification?

What is classification?

Structure of C*-algebras

- Elliott invariant $(\mathrm{Ell}(.))$
  Classifying C*-algebras using invariants $(\mathrm{K}$-theory, trace, ...)
  Examples: 1. A_θ (Elliott '93) (AT algebras),
  2. $A_\theta \rtimes F$, F finite cyclic (Echterhoff–Lück–Phillips–Walters '10) (AF algebras).

What is classification?

Structure of C*-algebras

- Elliott invariant $(\mathrm{Ell}(.))$
  Classifying C*-algebras using invariants $(\mathrm{K}$-theory, trace, ...)
  Examples: 1. A_θ (Elliott '93) (AT algebras),
  2. $A_\theta \rtimes F$, F finite cyclic (Echterhoff–Lück–Phillips–Walters '10) (AF algebras).

Isomorphism classes of C*-algebras

- $A_\theta \cong A_{\theta'}$ if and only if $\theta = \pm\theta' \pmod{\mathbb{Z}}$.
- A similar result holds for $A_\theta \rtimes F$.

What is classification?

Structure of C*-algebras

- Elliott invariant ($\mathrm{Ell}(.)$)
  Classifying C*-algebras using invariants (K-theory, trace, ...)
  Examples: 1. A_θ (Elliott '93) (AT algebras),
  2. $A_\theta \rtimes F$, F finite cyclic (Echterhoff–Lück–Phillips–Walters '10) (AF algebras).

Isomorphism classes of C*-algebras

- $A_\theta \cong A_{\theta'}$ if and only if $\theta = \pm\theta' \pmod{\mathbb{Z}}$.
- A similar result holds for $A_\theta \rtimes F$.

Question:

What happens for $A_\theta \rtimes \mathbb{Z}$?

Crossed product algebras

Crossed product algebras

Definition

$A \rtimes_\alpha G$, for a unital C*-algebra A and a discrete group G, has a natural representation (also called regular representation) ι on the Hilbert module $l^2(G, A)$ which is given by

$$\iota(a)(\xi)(g) = \alpha_{g^{-1}}(a)\xi(g), \quad \iota(h)(\xi)(g) = \xi(h^{-1}g),$$

for $a \in A$ and $g, h \in G$.

Some examples....

Some examples....

- (Rieffel) $A_\theta \cong C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$.

Some examples....

- (Rieffel) $A_\theta \cong C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$.
- (Watatani, Brenken) Define an action of $SL_2(\mathbb{Z})$ on A_θ by sending a matrix
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
to the automorphism α_A of A_θ defined by
$$\alpha_A(U_1) := e^{\pi i (ac)\theta} U_1^a U_2^c, \qquad \alpha_A(U_2) := e^{\pi i (bd)\theta} U_1^b U_2^d.$$

Some examples....

- (Rieffel) $A_\theta \cong C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$.
- (Watatani, Brenken) Define an action of $SL_2(\mathbb{Z})$ on A_θ by sending a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to the automorphism α_A of A_θ defined by

$$\alpha_A(U_1) := e^{\pi i(ac)\theta} U_1^a U_2^c, \qquad \alpha_A(U_2) := e^{\pi i(bd)\theta} U_1^b U_2^d.$$

Let $A \in SL_2(\mathbb{Z})$ be a matrix of infinite order, and consider the restriction of the above action to the (infinite cyclic) subgroup generated by A. Denote the corresponding crossed product by $\mathbb{Z}$ as $A_\theta \rtimes_A \mathbb{Z}$.

Some facts about A_θ

- A_θ is unital, simple, nuclear, monotracial. The trace is given by $\tau_\theta(\sum a_{mn} U^m V^n) := a_{00}$.

Some facts about A_θ

- A_θ is unital, simple, nuclear, monotracial. The trace is given by $\tau_\theta(\sum a_{mn}U^m V^n) := a_{00}$.
- A_θ has tracial rank zero. Also AT alegbra.

Some facts about A_θ

- A_θ is unital, simple, nuclear, monotracial. The trace is given by $\tau_\theta(\sum a_{mn} U^m V^n) := a_{00}$.
- A_θ has tracial rank zero. Also AT alegbra.
- $\mathrm{K}_0(A_\theta) \cong \mathbb{Z}^2$ with generators $[1]_0$ and $[p_\theta]_0$, where p_θ is a projection in A_θ satisfying $\tau_\theta(p_\theta) = \theta$ (this is the so-called *Rieffel projection*).

Some facts about A_θ

- A_θ is unital, simple, nuclear, monotracial. The trace is given by $\tau_\theta(\sum a_{mn}U^m V^n) := a_{00}$.
- A_θ has tracial rank zero. Also AT alegbra.
- $\mathrm{K}_0(A_\theta) \cong \mathbb{Z}^2$ with generators $[1]_0$ and $[p_\theta]_0$, where p_θ is a projection in A_θ satisfying $\tau_\theta(p_\theta) = \theta$ (this is the so-called *Rieffel projection*).
- $\mathrm{K}_1(A_\theta) \cong \mathbb{Z}^2$.

Some facts about A_θ

- A_θ is unital, simple, nuclear, monotracial. The trace is given by $\tau_\theta(\sum a_{mn} U^m V^n) := a_{00}$.
- A_θ has tracial rank zero. Also AT alegbra.
- $\mathrm{K}_0(A_\theta) \cong \mathbb{Z}^2$ with generators $[1]_0$ and $[p_\theta]_0$, where p_θ is a projection in A_θ satisfying $\tau_\theta(p_\theta) = \theta$ (this is the so-called *Rieffel projection*).
- $\mathrm{K}_1(A_\theta) \cong \mathbb{Z}^2$.
- $A_\theta \cong A_{\theta'}$ if and only if $\theta = \pm\theta' \pmod{\mathbb{Z}}$.

$A_\theta \rtimes_A \mathbb{Z}$: K-theory and generators

$A_\theta \rtimes_A \mathbb{Z}$: K-theory and generators

Theorem

1. *If* $\mathrm{Tr}(A) = 2$ *then* $I_2 - A^{-1}$ *has the "Smith normal form"* $\mathrm{diag}(h_1, 0)$*, and*

$$\begin{aligned} \mathrm{K}_0(A_\theta \rtimes_A \mathbb{Z}) &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \\ \mathrm{K}_1(A_\theta \rtimes_A \mathbb{Z}) &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{h_1}, \\ (\tau_A)_*(\mathrm{K}_0(A_\theta \rtimes_A \mathbb{Z})) &= \mathbb{Z} + \theta\mathbb{Z}, \\ \mathrm{K}_0(A_\theta \rtimes_A \mathbb{Z}) &= \langle [1]_0, i_*([p_\theta]_0), [P_A]_0 \rangle. \end{aligned}$$

$A_\theta \rtimes_A \mathbb{Z}$: K-theory and generators

Theorem

1. *If* $\mathrm{Tr}(A) = 2$ *then* $I_2 - A^{-1}$ *has the "Smith normal form"* $\mathrm{diag}(h_1, 0)$*, and*

$$\begin{aligned}
\mathrm{K}_0(A_\theta \rtimes_A \mathbb{Z}) &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \\
\mathrm{K}_1(A_\theta \rtimes_A \mathbb{Z}) &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{h_1}, \\
(\tau_A)_*(\mathrm{K}_0(A_\theta \rtimes_A \mathbb{Z})) &= \mathbb{Z} + \theta\mathbb{Z}, \\
\mathrm{K}_0(A_\theta \rtimes_A \mathbb{Z}) &= \langle [1]_0, i_*([p_\theta]_0), [P_A]_0 \rangle.
\end{aligned}$$

2. *If* $\mathrm{Tr}(A) \notin \{0, \pm 1, 2\}$ *then* $I_2 - A^{-1}$ *has the Smith normal form* $\mathrm{diag}(h_1, h_2)$*, and*

$$\begin{aligned}
\mathrm{K}_0(A_\theta \rtimes_A \mathbb{Z}) &\cong \mathbb{Z} \oplus \mathbb{Z}, \\
\mathrm{K}_1(A_\theta \rtimes_A \mathbb{Z}) &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{h_1} \oplus \mathbb{Z}_{h_2}, \\
(\tau_A)_*(\mathrm{K}_0(A_\theta \rtimes_A \mathbb{Z})) &= \mathbb{Z} + \theta\mathbb{Z}, \\
\mathrm{K}_0(A_\theta \rtimes_A \mathbb{Z}) &= \langle [1]_0, i_*([p_\theta]_0) \rangle.
\end{aligned}$$

$A_\theta \rtimes_A \mathbb{Z}$: sturucture

Theorem

- *The $\mathbb{Z}$-action on A_θ has the "tracial Rokhlin property".*

$A_\theta \rtimes_A \mathbb{Z}$: sturucture

Theorem

- *The $\mathbb{Z}$-action on A_θ has the "tracial Rokhlin property".*
- *$A_\theta \rtimes_A \mathbb{Z}$ is a unital, simple, nuclear, monotracial C^*-algebra with tracial rank zero and satisfies the UCT.*

$A_\theta \rtimes_A \mathbb{Z}$: sturucture

Theorem

- *The* $\mathbb{Z}$*-action on* A_θ *has the "tracial Rokhlin property".*
- $A_\theta \rtimes_A \mathbb{Z}$ *is a unital, simple, nuclear, monotracial* C^**-algebra with tracial rank zero and satisfies the UCT.*

Corollary

$A_\theta \rtimes_A \mathbb{Z} \cong A_{\theta'} \rtimes_B \mathbb{Z}$ *if and only if* $\mathrm{Ell}(A_\theta \rtimes_A \mathbb{Z})$ *and* $\mathrm{Ell}(A_{\theta'} \rtimes_B \mathbb{Z})$ *are isomorphic.*

$A_\theta \rtimes_A \mathbb{Z}$: classification

$A_\theta \rtimes_A \mathbb{Z}$: classification

Theorem

TFAE:

1. $A_\theta \rtimes_A \mathbb{Z}$ *and* $A_{\theta'} \rtimes_B \mathbb{Z}$ *are* $*$*-isomorphic;*
2. $\theta = \pm\theta' \pmod{\mathbb{Z}}$ *and* $P(I_2 - A^{-1})Q = I_2 - B^{-1}$ *for some* P, Q *in* $GL_2(\mathbb{Z})$*.*

$A_\theta \rtimes_A \mathbb{Z}$: classification

Theorem

TFAE:

1. $A_\theta \rtimes_A \mathbb{Z}$ *and* $A_{\theta'} \rtimes_B \mathbb{Z}$ *are* $*$*-isomorphic;*
2. $\theta = \pm\theta' \pmod{\mathbb{Z}}$ *and* $P(I_2 - A^{-1})Q = I_2 - B^{-1}$ *for some* P, Q *in* $GL_2(\mathbb{Z})$*.*

Corollary

Suppose $\mathrm{Tr}(A) = 3$*.* $\mathrm{K}_0(A_\theta \rtimes_A \mathbb{Z}) \cong \mathrm{K}_1(A_\theta \rtimes_A \mathbb{Z}) \cong \mathbb{Z}^2$*. The crossed product* $A_\theta \rtimes_A \mathbb{Z}$ *is an AT algebra and*

$$i_* : \mathrm{K}_0(A_\theta) \to \mathrm{K}_0(A_\theta \rtimes_A \mathbb{Z})$$

is an (order) isomorphism.

$A_\theta \rtimes_A \mathbb{Z}$: classification

Theorem

TFAE:

1. $A_\theta \rtimes_A \mathbb{Z}$ *and* $A_{\theta'} \rtimes_B \mathbb{Z}$ *are* $*$*-isomorphic;*
2. $\theta = \pm\theta' \pmod{\mathbb{Z}}$ *and* $P(I_2 - A^{-1})Q = I_2 - B^{-1}$ *for some* P, Q *in* $GL_2(\mathbb{Z})$*.*

Corollary

Suppose $\mathrm{Tr}(A) = 3$*.* $\mathrm{K}_0(A_\theta \rtimes_A \mathbb{Z}) \cong \mathrm{K}_1(A_\theta \rtimes_A \mathbb{Z}) \cong \mathbb{Z}^2$*. The crossed product* $A_\theta \rtimes_A \mathbb{Z}$ *is an AT algebra and*

$$i_* : \mathrm{K}_0(A_\theta) \to \mathrm{K}_0(A_\theta \rtimes_A \mathbb{Z})$$

is an (order) isomorphism. Hence $A_\theta \rtimes_A \mathbb{Z} \cong A_\theta$*.*

References

[1] C Bönicke, S Chakraborty, Z He, and H-C Liao. Isomorphism and morita equivalence classes for crossed products of irrational rotation algebras by cyclic subgroups of $SL_2(\mathbb{Z})$. *J. Funct. Anal.* 275 (2018), no. 11, 3208–3243.

[2] S. Echterhoff, W. Lück, N.Phillips, and S. Walters.The structure of crossed products of irrational rotation algebras by finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$. *J. Reine Angew. Math*, 639:173–221, 2010.

[3] G.A. Elliott and D.E. Evans. The structure of the irrational rotation C^*-algebra. *Ann. of Math. (2)*, 138(3):477–501, 1993.

Thank You