Single Commutators of Compact Operators

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Test question. Is every rank one projection operator a commutator of compact operators?

Breakthrough work by J. Anderson (1977)

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$$P = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

is a commutator of compact operators C and Z given by

$$C = \begin{pmatrix} 0 & A_1 & & \\ B_1 & 0 & A_2 & \\ & B_2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \qquad \text{and} \qquad Z = \begin{pmatrix} 0 & X_1 & & \\ Y_1 & 0 & X_2 & \\ & Y_2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

where

$$A_n = \begin{pmatrix} a_{1,n} & 0 & \cdots & 0 & 0 \\ 0 & a_{2,n} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & a_{n,n} & 0 \end{pmatrix} \qquad B_n = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -b_{1,n} & 0 & 0 & \cdots \\ 0 & -b_{2,n} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & \cdots & -b_{n,n} \end{pmatrix}$$

$$X_n = \begin{pmatrix} 0 & x_{1,n} & 0 & \cdots & 0 \\ 0 & 0 & x_{2,n} & 0 & 0 \\ \vdots & 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & x_{n,n} \end{pmatrix} \qquad Y_n = \begin{pmatrix} y_{1,n} & 0 & 0 & 0 \\ 0 & y_{2,n} & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & y_{n,n} \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$a_{k,n}(t) = (n+1-k)^t n^{-1}, \quad b_{k,n}(t) = k^t (n+1)^{-1}$$

$$x_{k,n}(t) = k^{1-t} n^{-1}, \quad y_{k,n}(t) = (n+1-k)^{(1-t)} (n+1)^{-1}$$

for each $0 < t < 1$.

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Question (Weiss '76): Are there any strictly positive operators that are commutators of compact operators?

P.-Weiss modification of Anderson's model (2013)

Compact positive diagonal operators with zero kernel as single commutator of compacts.

$$C = \begin{pmatrix} 0 & \sqrt{c_1}A_1 & & \\ \sqrt{c_1}B_1 & 0 & \sqrt{c_2}A_2 & \\ & \sqrt{c_2}B_2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, Z = \begin{pmatrix} 0 & \sqrt{c_1}X_1 & & \\ \sqrt{c_1}Y_1 & 0 & \sqrt{c_2}X_2 & \\ & \sqrt{c_2}Y_2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

The commutator $CZ - ZC$ is the diagonal operator

$$\operatorname{diag}(c_1, \frac{c_2 - c_1}{2}, \frac{c_2 - c_1}{2}, \cdots, \underbrace{\frac{c_{n+1} - c_n}{n+1}, \cdots, \frac{c_{n+1} - c_n}{n+1}}_{n+1 \text{ times}}, \cdots)$$

To keep C, Z compact operators, choose $\frac{c_n}{n} \to 0$, and furthermore to obtain a strictly positive compact operator choose $c_n \uparrow$.

Another variation of the Anderson model (P. -Petrovic-Weiss (2018)):

Positive compact diagonal with 'distinct diagonal entries' as single commutator of compacts

- The sequence (d_n) of positive numbers is increasing
- $\lim_{n\to\infty} \frac{d_n}{n} = 0$
- $\liminf \ n\left(\frac{d_{n+1}}{d_n} - 1\right) > 0$
- $f : \mathbb{N} \to \mathbb{R}$ such that $\max_{1\le k\le n+1} \frac{f(k)}{\sqrt{n}} \to 0$ as $n \to \infty$

For $n \in \mathbb{N}$ and $1 \le k \le n$, we define numbers

$$a_{k,n} = \sqrt{d_n}\frac{\sqrt{n+1-k}}{n}, \quad x_{k,n} = \sqrt{d_n}\frac{f(k)}{n},$$
$$b_{k,n} = \sqrt{d_n}\frac{f(k)}{n+1}, \quad y_{k,n} = \sqrt{d_n}\frac{\sqrt{n+1-k}}{n+1},$$

Then $CZ - ZC$ is a strictly positive compact operator with distinct entries.

Limitation of the Anderson Model

A_n, B_n, X_n, and Y_n denote arbitrary rectangular matrices of size $n \times (n+1), (n+1) \times n, n \times (n+1)$, and $(n+1) \times n$ respectively in the Anderson model.

Consider the Anderson operators C and Z with these A_n, B_n, X_n, and Y_n.

$$C = \begin{pmatrix} 0 & A_1 & & \\ B_1 & 0 & A_2 & \\ & B_2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 0 & X_1 & & \\ Y_1 & 0 & X_2 & \\ & Y_2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

Then there are strictly positive compact diagonal operators $D = \operatorname{diag}(d_n)$ which cannot be obtained using the Anderson model with $C, Z \in K(H)$, i.e., $CZ - ZC \neq D$.

A sufficient condition for nonsolvability is

$$\frac{1}{n}\left(d_1 + \cdots + d_{\frac{(n+2)(n+1)}{2}}\right) \to \infty.$$

Example: $d_n = \frac{1}{log(n+1)}$

Suppose $CZ - ZC = D$ where $D = \mathrm{diag}(d_n)$ such that $0 < d_n$ with $d_n \to 0$ and satisfying

$$\frac{1}{n}\left(d_1 + \cdots + d_{\frac{(n+2)(n+1)}{2}}\right) \to \infty$$

and C and Z compact operators.

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and C and Z compact operators. The first $(n+1) \times (n+1)$ diagonal blocks of $CZ - ZC = D$ are:

$$A_1 Y_1 - X_1 B_1 = D_1$$
$$\vdots$$
$$B_n X_n - Y_n A_n + A_{n+1} Y_{n+1} - X_{n+1} B_{n+1} = D_{(n+1)\times(n+1)}$$

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$$tr(A_{n+1} Y_{n+1}) - tr(X_{n+1} B_{n+1}) = d_1 + d_2 + \cdots + d_{\frac{(n+2)(n+1)}{2}}$$

$$\begin{aligned} d_1 + d_2 + \cdots + d_{\frac{(n+2)(n+1)}{2}} &= |tr(A_{n+1} Y_{n+1}) - tr(X_{n+1} B_{n+1})| \\ &\leq (n+1)(\|A_{n+1}\|\|Y_{n+1}\| + \|X_{n+1}\|\|B_{n+1}\|) \end{aligned}$$

$$\frac{1}{n+1}\left(d_1 + \cdots + d_{\frac{(n+2)(n+1)}{2}}\right) \leq \|A_{n+1}\|\|Y_{n+1}\| + \|B_{n+1}\|\|X_{n+1}\|$$

Suppose $CZ - ZC = D$ where $D = \text{diag}(d_n)$ such that $0 < d_n$ with $d_n \to 0$ and satisfying

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$$\frac{1}{n+1}\left(d_1 + \cdots + d_{\frac{(n+2)(n+1)}{2}}\right) \leq \|A_{n+1}\|\|Y_{n+1}\| + \|B_{n+1}\|\|X_{n+1}\|$$

Therefore, the sequence

$$\frac{1}{n+1}\left(d_1 + d_2 + \cdots + d_{\frac{(n+2)(n+1)}{2}}\right) \text{ is bounded.}$$

A new model that avoids Anderson model constraint: Staircase form

For $T \in B(H)$, there is a tri-block diagonal partition of the matrix of T with respect to an orthonormal basis given by

$$T = \begin{pmatrix} C_1 & A_1 & 0 & \\ B_1 & C_2 & A_2 & \\ 0 & B_2 & C_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

where C_1, A_1, and B_1 are 1×1, 1×2 and 2×1 matrices respectively. And for $k \geq 1$,

$$C_{k+1} : 2(3^{k-1}) \times 2(3^{k-1}) \text{ matrix}$$

$$A_{k+1} : 2(3^{k-1}) \times 2(3^{k}) \text{ matrix}$$

$$B_{k+1} : 2(3^{k}) \times 2(3^{k-1}) \text{ matrix}$$

$$C = \begin{pmatrix} 0 & A_1 & & \\ B_1 & 0 & A_2 & \\ & B_2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 0 & X_1 & & \\ Y_1 & 0 & X_2 & \\ & Y_2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

where C and Z are in the staircase form.

Suppose $CZ - ZC = D$ where $D = \text{diag}(d_n)$ with $d_n \downarrow 0$.

$$\frac{1}{2(3^n)}(d_1 + \cdots + d_{3^n}) \leq \|A_{n+1}\|\|Y_{n+1}\| + \|X_{n+1}\|\|B_{n+1}\|$$

L.H.S. and R.H.S. tend to 0, if C and Z are assumed compact. This is in contrast to the Anderson model that yields

$$\frac{1}{n+1}\left(d_1 + \cdots + d_{\frac{(n+2)(n+1)}{2}}\right) \leq \|A_{n+1}\|\|Y_{n+1}\| + \|B_{n+1}\|\|X_{n+1}\|$$

for which there are compact diagonals D for which $CZ - ZC \neq D$ for C and Z bounded operators.

Thank you ☺