Weighted join operators on rooted directed trees

Sameer Chavan

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Abstract

We capitalize on the order structure of directed trees to introduce and study the classes of weighted join operators and their rank one extensions. In particular, we discuss the issue of closedness, unravel the structure of Hilbert space adjoint and identify various spectral parts of members of these classes. Certain discrete Hilbert transforms arise naturally in the spectral theory of rank one extensions of weighted join operators.

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We capitalize on the order structure of directed trees to introduce and study the classes of weighted join operators and their rank one extensions. In particular, we discuss the issue of closedness, unravel the structure of Hilbert space adjoint and identify various spectral parts of members of these classes. Certain discrete Hilbert transforms arise naturally in the spectral theory of rank one extensions of weighted join operators.

This is a joint work with R. Gupta and K. B. Sinha.

Spiral-like ordering (SLO)

- V is countably infinite
- $\mathcal{T} = (V, E)$ a rooted directed tree with root root
- $\mathrm{Chi}(\cdot)$ as a function from the power set $\mathcal{P}(V)$ into itself
- $\mathrm{Chi}^{\langle k \rangle}(\cdot)$ is $\mathrm{Chi}(\cdot)$ composed k-times with itself
- The *depth* of $u \in V$ is the unique non-negative integer d_u such that $u \in \mathrm{Chi}^{\langle \mathrm{d}_u \rangle}(\mathrm{root})$.

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Definition

Define the relation $\leqslant$ on V as follows:

$$v \ \leqslant \ w \ \text{ if } \ \mathrm{d}_v \ \leqslant \ \mathrm{d}_w.$$

- V is a partially ordered set: $\leqslant$ is reflexive and transitive.
- Given $v, w \in V$, $\exists\ u \in V$ such that $v \leqslant u$ and $w \leqslant u$.

Extended directed tree

Definition

Let $\mathcal{T} = (V, E)$ be a rooted directed tree. The *extended directed tree* $\mathcal{T}_\infty$ *associated with* $\mathcal{T}$ is the directed graph (V_∞, E_∞) given by

$$V_\infty = V \sqcup \{\infty\}, \quad E_\infty = E \sqcup \{(u, \infty) : u \in V\}.$$

Remark. ∞ is a boundary point when $\mathcal{T}_\infty$ is considered as a directed graph with boundary.

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- [A.R. Pruss, 1988, Duke Math J]
- [T. Biyikoğlu and J. Leydold, 2007, J. Combin. Theory Ser. B]
- [J. Friedman, 1993, Duke Math J]

Join and meet operations

Definition (Join operation)

For $u, v \in V_\infty$, set

$$u \sqcup v = \begin{cases} u & \text{if } u \in \mathrm{Des}(v), \\ v & \text{if } v \in \mathrm{Des}(u), \\ \infty & \text{otherwise.} \end{cases}$$

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Definition (Meet operation)

Let $u, v \in V$. We say that u *meets* v if there exists a unique vertex $\omega \in V$ such that

$$\sup_{w \in \mathfrak{par}(u,v)} \mathrm{d}_w = \mathrm{d}_\omega, \quad \text{where}$$

$$\mathfrak{par}(u, v) := \{w \in V : \mathrm{par}^{\langle n \rangle}(u) = w = \mathrm{par}^{\langle m \rangle}(v) \text{ for some } m, n \in \mathbb{N}\}.$$

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- Any two vertices always meet.

Definition (Join operation at a base point)

Fix $\mathrm{b} \in V_\infty$ (*base point*) and let $u, v \in V_\infty$.

The binary operation $\sqcup_\mathrm{b}$ on V_∞ is given by

$$u \sqcup_\mathrm{b} v = \begin{cases} u \sqcap v & \text{if } u, v \in \mathrm{Asc}(\mathrm{b}), \\ u & \text{if } v = \mathrm{b}, \\ v & \text{if } \mathrm{b} = u, \\ u \sqcup v & \text{otherwise.} \end{cases}$$

- Note that $\sqcup_\mathrm{root} = \sqcup$ and $\sqcup_\infty = \sqcap$.

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- Note that $\sqcup_{\mathrm{root}} = \sqcup$ and $\sqcup_\infty = \sqcap$.

Theorem

For every $\mathrm{b} \in V$, *the pair* $(V_\infty, \sqcup_{\mathrm{b}})$ *is a commutative semigroup admitting* b *as a neutral element and* ∞ *as an absorbing element.*

Weighted join operators

$u \in V$, $\mathsf{b} \in V_\infty$, $\lambda_u = \{\lambda_{uv}\}_{v \in V_\infty} \subseteq \mathbb{C}$ with $\lambda_{u\infty} = 0$.

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The *weighted join operator* $W_{b,u}$ *on* $\mathcal{T} = (V, E)$ by

$$\mathcal{D}(W_u^{(b)}) := \{f \in \ell^2(V) \colon \Lambda_u^{(\mathrm{b})} f \in \ell^2(V)\},$$
$$W_u^{(b)} f := \Lambda_u^{(\mathrm{b})} f, \quad f \in \mathcal{D}(Wb, u),$$

where $\Lambda_u^{(\mathrm{b})}$ is given by

$$(\Lambda_u^{(\mathrm{b})} f)(w) := \sum_{v \in M_u^{(\mathrm{b})}(w)} \lambda_{uv} f(v), \quad w \in V,$$
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- Well-definedness and size of $M_u^{(\mathrm{b})}(w)$
- Cf. [Z. Jablonski, Il B. Jung, J. Stochel, Memoirs AMS, 2012]

For $\mathsf{b}, u \in V$, consider the subspace $\ell^2(U_u^{(\mathsf{b})})$ of $\ell^2(V)$, where

$$U_u^{(\mathsf{b})} = \begin{cases} V \setminus \{u\} & \text{if } \mathsf{b} = u, \\ \mathrm{Asc}(u) \cup \mathrm{Des}_b(u) & \text{if } \mathsf{b} \in \mathrm{Des}_u(u), \\ \mathrm{Des}_u(u) & \text{otherwise.} \end{cases}$$

Here $\mathrm{Des}_v(u) = \mathrm{Des}(u) \setminus [u, v]$, $v \in \mathrm{Des}(u)$.

- $\{e_v\}_{v \in U_u^{(\mathsf{b})}}$: standard orthonormal basis of $\ell^2(U_u^{(\mathsf{b})})$
- diagonal operator in $\ell^2(U_u^{(\mathsf{b})})$:

$$D_u^{(\mathsf{b})} e_v = \lambda_{uv} e_v, \quad v \in U_u^{(\mathsf{b})}.$$

- the rank one operator $N_u^{(\mathsf{b})} = e_u \otimes e_{A_u}$ in $\ell^2(V \setminus U_u^{(\mathsf{b})})$, where $e_{A_u} := \sum_{v \in A_u} \lambda_{uv} e_v$ and the subset A_u of V is given by

$$A_u = \begin{cases} [u, \mathsf{b}] & \text{if } \mathsf{b} \in \mathrm{Des}(u), \\ \mathrm{Asc}(u) \cup \{\mathsf{b}, u\} & \text{otherwise.} \end{cases}$$

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Theorem

$W_u^{(\mathrm{b})}$ is unitarily equivalent to $D_u^{(\mathrm{b})} \oplus N_u^{(\mathrm{b})}$.

Rank one extensions

- $f \in \ell^2(V \setminus U_u^{(\mathrm{b})})$, $\underline{g \notin \ell^2(U_u^{(\mathrm{b})})}$
- the rank one operator $f \oslash g$ is given by

$$\mathcal{D}(f \oslash g) = \Big\{ h \in \ell^2(U_u^{(\mathrm{b})}) : \sum_{j \in U_u^{(\mathrm{b})}} h(j)\overline{g(j)} \text{ is convergent} \Big\},$$

$$f \oslash g(h) = \Big(\sum_{j \in U_u^{(\mathrm{b})}}^{\infty} h(j)\overline{g(j)} \Big) f, \quad h \in \mathcal{D}(f \oslash g).$$

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Definition

The rank one extension $W_{f,g}$ of $W_u^{(b)}$ is given by

$$\begin{aligned} \mathcal{D}(W_{f,g}) &= \{(h,k) : h \in \mathcal{D}(D_u^{(\mathrm{b})}) \cap \mathcal{D}(f \otimes g),\ k \in \ell^2(V \setminus U_u^{(\mathrm{b})})\} \\ W_{f,g} &= \begin{bmatrix} D_u^{(\mathrm{b})} & 0 \\ f \oslash g & N_u^{(\mathrm{b})} \end{bmatrix}. \end{aligned}$$

- Spectral theory of block operator matrices:
  - [C. Tretter, Spectral theory of block operator matrices, 2008]
  - [A. Batkai, P. Binding, A. Dijksma, R. Hryniv and H. Langer, 2005, Math. Nachr.]
  - [M. Möller and F. H. Szafraniec, Proc AMS, 2008]
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  - [M. Möller and F. H. Szafraniec, Proc AMS, 2008]
  - [C. Tretter, J. Funct A, 2009]
- Rank one perturbations of self-adjoint operators:
  - [E. Ionascu, Inte Eq Oper Th, 2001]
  - [C. Foias, Il B. Jung, E. Ko and C. Pearcy, J. Funct A, 2007]
  - [A. Baranov and D. Yakubovich, Advances in Math, 2016]
  - [C. Liaw and S. Treil, J. Funct A, 2009]

An illustration

- $\lambda_{uv} := \mathsf{d}_v - \mathsf{d}_u,\ v \in U_u^{(\mathsf{b})}$
- For $w \in V \setminus U_u^{(\mathsf{b})}$, $x \in \mathbb{R}$, let $f = e_w,\ g_x = \sum_{v \in U_u^{(\mathsf{b})}} \mathsf{d}_v^x e_v.$

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Theorem

Assume that $(\mathrm{Des}(u), E_u)$ is a countably infinite narrow subtree of $\mathcal{T}$. Then $\sigma(W_{f,g_x}) \subsetneq \mathbb{C}$ iff $x < 1/2$. Under latter condition, TFAT:

(i) *W_{f,g_x} is a closed operator with domain $\mathcal{D}(D_u^{(\mathsf{b})}) \oplus \ell^2(V \setminus U_u^{(\mathsf{b})})$.*

(ii) $\sigma(W_{f,g_x}) = \{\mathsf{d}_v - \mathsf{d}_u : v \in U_u^{(\mathsf{b})} \cup \{u\}\} = \sigma_p(W_{f,g_x})$.

(iii) $\sigma_e(W_{f,g_x}) \setminus \{0\} = \Big\{\mathsf{d}_v - \mathsf{d}_u : v \in U_u^{(\mathsf{b})},\ |\mathsf{Chi}^{\langle \mathsf{d}_v \rangle}(\mathsf{root})| = \infty\Big\}$. *Moreover, $ind_{W_{f,g_x}} = 0$ on $\mathbb{C} \setminus \sigma_e(W_{f,g_x})$.*

(iv) *W_{f,g_x} is a sectorial operator, which generates a strongly continuous quasi-bounded semigroup.*

(v) *W_{f,g_x} is never normal.*

(vi) *If, in addition, $\mathcal{T}$ is leafless, then W_{f,g_x} admits a compact resolvent if and only if the set $V_{\prec} \cap \mathrm{Asc}(u) = \emptyset$.*

Compatibility conditions

- $\operatorname{dist}(\mu, \lambda_u) = \inf\left\{|\mu - \lambda_{uv}| : v \in \operatorname{supp}\ell^2(U_u^{(\mathrm{b})})\right\},\ \mu \in \mathbb{C}$
- $\Gamma_{\lambda_u} = \left\{\mu \in \mathbb{C} : \operatorname{dist}(\mu, \lambda_u) > 0\right\}.$

Compatibility conditions

- $\mathrm{dist}(\mu, \lambda_u) = \inf\left\{|\mu - \lambda_{uv}| : v \in \operatorname{supp} \ell^2(U_u^{(b)})\right\}, \ \mu \in \mathbb{C}$
- $\Gamma_{\lambda_u} = \left\{\mu \in \mathbb{C} : \mathrm{dist}(\mu, \lambda_u) > 0\right\}.$

Definition

(1) We say that $W_{f,g}$ *satisfies compatibility condition I* if there exists $\mu_0 \in \Gamma_{\lambda_u}$ such that $g_{\lambda_u,\mu_0} \in \ell^2(U_u^{(b)})$, where

$$g_{\lambda_u,\mu_0}(v) := \frac{g(v)}{\lambda_{uv} - \mu_0}, \quad v \in \operatorname{supp} \ell^2(U_u^{(b)}).$$

(2) We say that $W_{f,g}$ *satisfies compatibility condition II* if

$$\sum_{v \in \operatorname{supp} \ell^2(U_u^{(b)})} \frac{|g(v)|^2}{|\lambda_{uv}|^2 + 1} < \infty.$$

If (1) or (2) holds, then $W_{f,g}$ *satisfies a compatibility condition*.

Theorem

If $W_{f,g}$ satisfies a compatibility condition, then we have the domain inclusion $\mathcal{D}(D_u^{(b)}) \subseteq \mathcal{D}(f \oslash g)$. If $\Gamma_{\lambda_u} \neq \emptyset$, TFAE:

(i) $\mathcal{D}(D_u^{(b)}) \subseteq \mathcal{D}(f \oslash g)$.

(ii) *$W_{f,g}$ satisfies compatibility condition I.*

(iii) *The discrete Hilbert transform $H_{\lambda_u,g}$ given by*

$$H_{\lambda_u,g}(h) \; = \sum_{v \in U_u^{(\mathrm{b})}} \frac{h(v)\overline{g(v)}}{\mu - \lambda_{uv}}$$

is well-defined for every $\mu \in \Gamma_{\lambda_u}$ and every $h \in \ell^2(U_u^{(\mathrm{b})})$.

(iv) *For every $\mu \in \mathit{\Gamma}_{\lambda_u}$, $L_{\lambda_u,\mu} := (f \oslash g)(D_u^{(b)} - \mu)^{-1}$ defines a bounded linear transformation from $\ell^2(U_u^{(\mathrm{b})})$ into $\ell^2(V \setminus U_u^{(\mathrm{b})})$.*

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- *The operator $L_{\lambda_u,\mu}$ appears in the Frobenius-Schur-type factorization in [F. Atkinson et al, 1994, Math. Nachr, Eq (1.6)].*

Spectral picture

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Theorem

- *$\sigma(W_{f,g})$ is a proper closed subset of $\mathbb{C}$ if and only if $W_{f,g}$ satisfies the compatibility condition I.*

- *In case $W_{f,g}$ satisfies the compatibility condition I*
    - $\sigma(W_{f,g}) = \overline{\sigma_p(W_{f,g})}$.
    - $\pi(W_{f,g}) = \mathbb{C} \setminus \overline{\sigma_p(W_{f,g})}$.
- *In case $W_{f,g}$ does not satisfy the compatibility condition I*
    - $\sigma(W_{f,g}) = \mathbb{C}$.
    - *Either $W_{f,g}$ is not closed or $\pi(W_{f,g}) = \emptyset$.*

• In general, the spectrum of $W_{f,g}$ may not be the topological closure of its point spectrum.

Operator-sum and form-sum

Theorem

Let J be a countably infinite directed set and let D_λ be a sectorial operator in $\ell^2(J)$ and let $f \in \ell^2(J)$. Let $g : J \to \mathbb{C}$ be such that for some $z_0 \in \rho(D_\lambda)$,

$$\sum_{j \in J} \frac{|g(j)|^2}{|\lambda_j - z_0|^2} \ < \ \infty.$$

Then $D_\lambda + f \oslash g$ is sectorial in $\ell^2(J)$ with domain $\mathcal{D}(D_\lambda)$.

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- Consider a sectorial diagonal operator D_λ in $\ell^2(J)$
- R_λ be the unique square-root of D_λ with $\mathcal{D}(R_\lambda) = \mathcal{D}(R_\lambda^*)$
- Consider the form Q_R given by

$$Q_R(h, k) \; := \; \langle R_\lambda h, \, R_\lambda^* k \rangle, \quad h, k \in \mathcal{D}(R_\lambda).$$

Theorem

- D_λ *be a sectorial diagonal operator in* $\ell^2(J)$ *with angle* $\theta \in (0, \pi/2)$ *and vertex* 0
- $f : J \to \mathbb{C}$ *and* $g : J \to \mathbb{C}$ *be such that for some* $z_0 \in (-\infty, 0)$,

$$\sum_{j\in J} \frac{|f(j)|^2}{|\sqrt{\lambda_j} - z_0|^2} \ < \ \infty, \quad \sum_{j\in J} \frac{|g(j)|^2}{|\sqrt{\lambda_j} - z_0|^2} \ < \ \infty.$$

Then $Q_{f,g}(h,k) := \sum_{j\in J} h(j)\overline{g(j)} \ \sum_{j\in J} f(j)\overline{k(j)}$ *is defined for all* $h, k \in \mathcal{D}(R_\lambda)$. *Moreover, the form* $Q_R + Q_{f,g}$ *is sectorial and there exists a unique sectorial operator* T *in* $\ell^2(J)$ *with domain contained in the domain of* Q_R *such that*

$$Q_R(h,k) \ + \ Q_{f,g}(h,k) \ = \ \langle Th, \ k\rangle, \quad h \in \mathcal{D}(T), \ k \in \mathcal{D}(Q_R).$$

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- Rootless case