Annular representations of free product categories

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Overview

$\mathcal{C}$, $\mathcal{D}$ - rigid semi-simple C*-tensor categories with simple tensor units

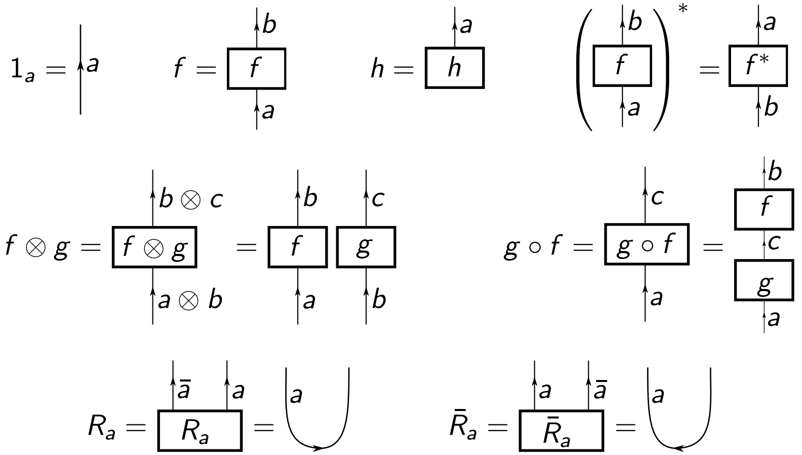
- $\mathcal{C} * \mathcal{D}$
- A description of $Rep(\mathcal{A}(\mathcal{C} * \mathcal{D}))$- annular representation category of $\mathcal{C} * \mathcal{D}$

★ Free product already appeared in the work of Bisch- V Jones (in the context of subfactors) and Wang (in the context of compact quantum groups).

Preliminaries- Graphical calculus for $*$-tensor categories

$\mathcal{C}$ - a rigid $*$-tensor category

$a, b, c \in \mathrm{Obj}(\mathcal{C}),\ f \in \mathcal{C}(a,b),\ g \in \mathcal{C}(b,c),\ h \in \mathcal{C}(\mathbb{1}, a)$

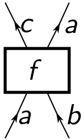


Preliminaries- Annular algebra of a category

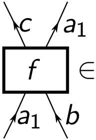
$\mathcal{C}$ - rigid semi-simple strict C^*-tensor category with simple tensor unit $\mathbb{1}$.

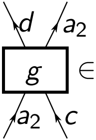
- Let $\mathbb{1} \in \Lambda \subseteq \mathrm{Obj}(\mathcal{C})$
- $\mathrm{Irr}(\mathcal{C})$ be a set of representatives of isomorphism classes of simple objects in $\mathcal{C}$ with $\mathbb{1} \in \mathrm{Irr}(\mathcal{C})$
- For any $a \in \mathrm{Irr}(\mathcal{C})$ and $b \in \mathrm{Obj}(\mathcal{C})$, there is a natural inner product on $\mathcal{C}(a, b)$ given by $g^* f = \langle f, g \rangle 1_a$, for $f, g \in \mathcal{C}(a, b)$
- **Annular algebra of $\mathcal{C}$ with weight set Λ**, $\mathcal{A}\Lambda$, is defined as a vector space by

$$\mathcal{A}\Lambda := \bigoplus_{b,c \in \Lambda,\ a \in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(a \otimes b, c \otimes a)$$

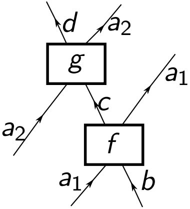
- $\mathcal{A}\Lambda^a_{b,c} := \mathcal{C}(a \otimes b, c \otimes a) \ni f \rightsquigarrow$ 

Preliminaries- Annular algebra of a category

- 

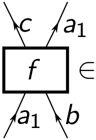
$\in \mathcal{A}\Lambda^{a_1}_{b,c}$, 

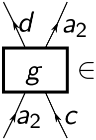
$\in \mathcal{A}\Lambda^{a_2}_{c,d}$, $a \in \mathrm{Irr}(\mathcal{C})$,



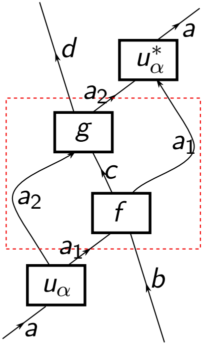
$\in \mathcal{C}\left((a_2 \otimes a_1) \otimes b, d \otimes (a_2 \otimes a_1)\right)$

- $\{u_\alpha\}_\alpha \in onb\left(\mathcal{C}(a, (a_2 \otimes a_1))\right)$

- 

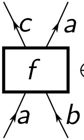
$\in \mathcal{A}\Lambda^{a_1}_{b,c}$, 

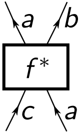
$\in \mathcal{A}\Lambda^{a_2}_{c,d}$, $a \in \mathrm{Irr}(\mathcal{C})$,

$(g.f)_a :=$ 

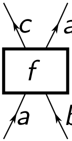
$\in \mathcal{C}\left(a \otimes b, d \otimes a\right) = \mathcal{A}\Lambda^{a}_{b,d}$

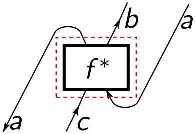
- $\{u_\alpha\}_\alpha \in onb\left(\mathcal{C}(a,(a_2 \otimes a_1))\right)$

- For 

$\in \mathcal{A}\Lambda^{a}_{b,c}$, $f^{\#} :=$ 

$\in \mathcal{C}(c \otimes a, a \otimes b)$

- For 

$\in \mathcal{A}\Lambda^{a}_{b,c}$, $f^{\#} :=$ 

$\in \mathcal{A}\Lambda^{\bar{a}}_{c,b}$
- With this product and $\#$, $\mathcal{A}\Lambda$ becomes an associative $*$-algebra
- **(Ocneanu's) Tube algebra** $\mathcal{AC}$ is the annular algebra with weight set $\mathrm{Irr}(\mathcal{C})$
- For any weight set Λ, the **annular representation category** $Rep(\mathcal{A}\Lambda)$ is simply the category of (non-degenerate) $*$-representations of $\mathcal{A}\Lambda$ as bounded operators on a Hilbert space. It is W*-category.

Preliminaries- Annular representation category

- When $\Lambda = \text{Obj}(\mathcal{C})$, $Rep(\mathcal{A}\Lambda) \cong \mathcal{Z}(\text{Ind-}\mathcal{C})$, which appears in the work of Neshveyev-Yamashita
- A weight set Λ is said to be **full** if every simple object is equivalent to a sub-object of some $b \in \Lambda$

Theorem (Ghosh-C Jones, 2016)

If Λ is full, then $Rep(\mathcal{A}\Lambda) \cong Rep(\mathcal{A}\mathcal{C})$

Preliminaries- Annular representation category

- **An approach** - look at representations of the unital centralizer algebras $\mathcal{A}\Lambda_{b,b}$ and see which of them extend to whole of $\mathcal{A}\Lambda$ ("weight b admissible representations")
- $C_u^*(\mathcal{A}\Lambda_{b,b})$- universal C*-completion with respect to weight b admissible representations
- A representation of $\mathcal{A}\Lambda_{b,b}$ is admissible if and only if it extends to a representation of $C_u^*(\mathcal{A}\Lambda_{b,b})$
- $\mathrm{Fus}(\mathcal{C}) := \mathbb{C}[\mathrm{Irr}(\mathcal{C})] \cong \mathcal{A}\Lambda_{\mathbb{1},\mathbb{1}}$
- $C_u^*(\mathcal{C}) := C_u^*(\mathrm{Fus}(\mathcal{C}))$

Why annular algebra and annular representations ?

Direction-1:

- Extremal subfactor $N \subseteq M \leftrightarrow P^{N \subseteq M}$ "subfactor planar algebra"
- V Jones introduced the notion of "affine annular category of a planar algebra P", denoted by $\mathcal{A}P$
- $Rep(\mathcal{A}P)$, the category of $*$-linear functors from $\mathcal{A}P$ into the category of vector spaces with morphisms as natural transformations
- $N \subseteq M \rightsquigarrow \mathcal{C}_{NN}$ and $\mathcal{C}_{MM}$, the categories of $N - N$ and $M - M$ bimodules appearing as submodules of tensor powers of ${}_N L^2(M) \underset{M}{\otimes} \overline{L^2(M)}_N$ and ${}_M \overline{L^2(M)} \underset{N}{\otimes} L^2(M)_M$ respectively
- $Rep(\mathcal{A}P^{N \subseteq M}) \cong Rep(\mathcal{AC}_{NN}) \cong Rep(\mathcal{AC}_{MM})$

Why annular algebra and annular representations ?

Direction-2:

• Popa-Vaes introduced the concept of cp-multipliers for $\mathcal{C}$ which are a class of functions in $l^\infty(\mathrm{Irr}(\mathcal{C}))$.

• cp-multipliers give positive linear functionals on the fusion algebra $\mathbb{C}[\mathrm{Irr}(\mathcal{C})]$ after a certain normalization.

• A $*$-representation is said to be **admissible** if every vector state is a normalization of some cp-multiplier.

• The class of admissible representations provides a good notion for the representation theory for $\mathcal{C}$

• Popa-Vaes' admissibility $\cong$ Ghosh-Jones' weight $\mathbb{1}$ admissibility

• Annular representation theory encapsulates the representation theory of Popa-Vaes

• This correspondence also allows to state properties like amenability, Haagerup and property (T) for C*-tensor categories in terms of convergence of certain states on the fusion algebra

Preliminaries- Free product of categories

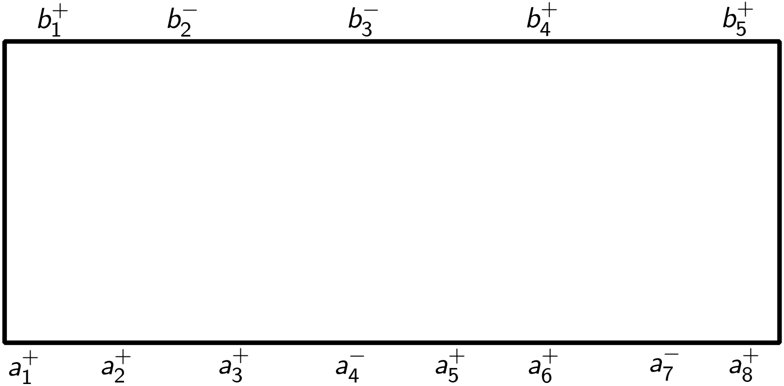
- $\mathcal{C}_+$ and $\mathcal{C}_-$ be two strict semi-simple C*-tensor categories with simple tensor units $\mathbb{1}_+$ and $\mathbb{1}_-$ respectively
- Σ- words with letter in $\mathrm{Obj}(\mathcal{C}_+) \cup \mathrm{Obj}(\mathcal{C}_-)$
- $\sigma \in \Sigma \rightsquigarrow (\sigma_+, \sigma_-) \rightsquigarrow (t(\sigma_+), t(\sigma_-)) \in \mathrm{Obj}(\mathcal{C}_+) \times \mathrm{Obj}(\mathcal{C}_-)$
- For $\sigma = a_1^+ a_2^- a_3^+ a_4^- a_5^-$, $a_i^\varepsilon \in \mathrm{Obj}(\mathcal{C}_\varepsilon)$, $\varepsilon \in \{+, -\}$

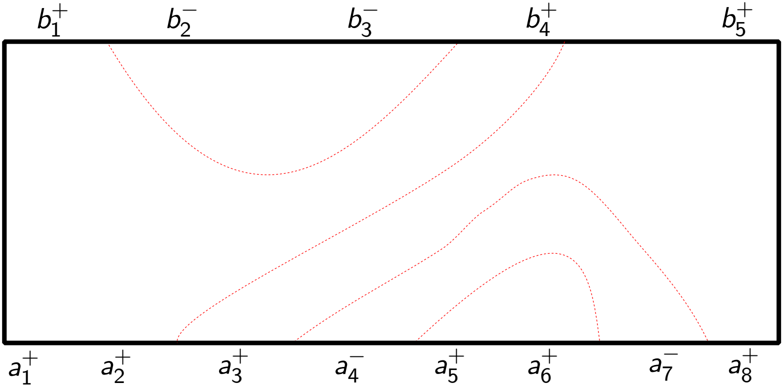
  $(\sigma_+, \sigma_-) = (a_1^+ a_3^+, a_2^- a_4^- a_5^-)$

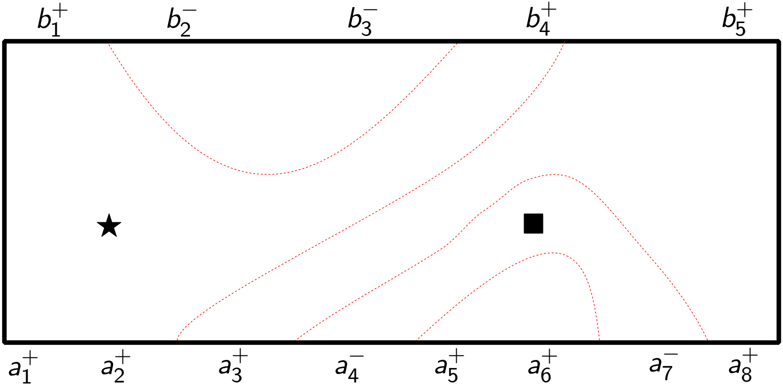
  $(t(\sigma_+), t(\sigma_-)) = (a_1^+ \otimes a_3^+, a_2^- \otimes a_4^- \otimes a_5^-)$

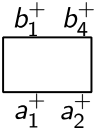
Preliminaries- Free product of categories

- $\sigma = a_1^+ a_2^+ a_3^+ a_4^- a_5^+ a_6^+ a_7^- a_8^+$ and $\tau = b_1^+ b_2^- b_3^- b_4^+ b_5^+$ with $a_i^\varepsilon, b_j^\varepsilon \in \mathcal{C}_\varepsilon$, $\varepsilon \in \{+, -\}$
- A '(σ, τ)-*NCP*' consists of:

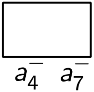



b_1^+ b_2^- b_3^- b_4^+ b_5^+
a_1^+ a_2^+ a_3^+ a_4^- a_5^+ a_6^+ a_7^- a_8^+

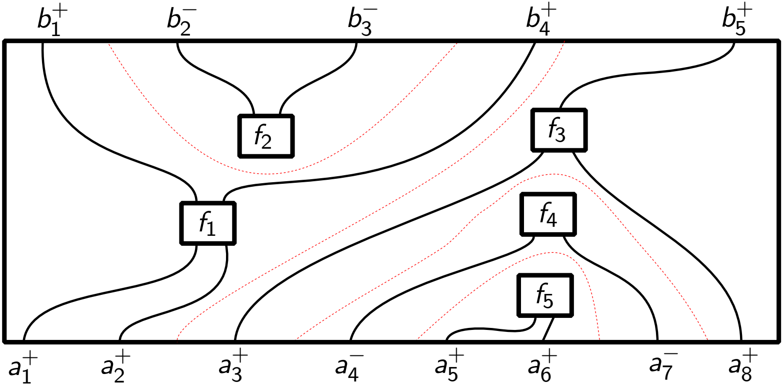


★ $\rightsquigarrow$ 

$\rightsquigarrow a_1^+ \otimes a_2^+ \to b_1^+ \otimes b_4^+$ in $\mathrm{Hom}(\mathcal{C}_+)$

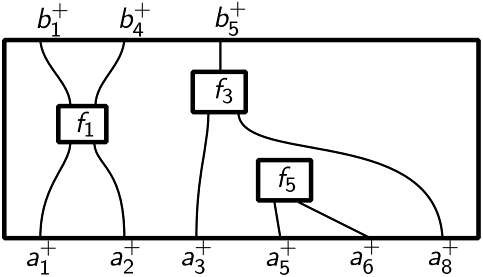
■ $\rightsquigarrow$ 

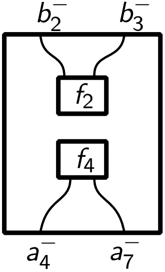
$\rightsquigarrow a_4^- \otimes a_7^- \to \mathbb{1}$ in $\mathrm{Hom}(\mathcal{C}_-)$



- (σ, τ)-NCP comes with the data of partition as well as the set of morphisms assigned for each non-crossing partition.
- $NCP(\sigma, \tau) := \{T : T \text{ is a } (\sigma, \tau)\text{-NCP}\}$

- $T \in NCP(\sigma, \tau) \overset{\text{uniquely}}{\rightsquigarrow} T_{\pm} \in NCP(\sigma_{\pm}, \tau_{\pm})$

$T_+ =$ 

$T_- =$ 

Preliminaries- Free product of categories

- $T_{\pm} \rightsquigarrow Z_{T_{\pm}} \in \mathcal{C}_{\pm}\left(t(\sigma_{\pm}), t(\tau_{\pm})\right)$ using the standard graphical calculus for tensor categories

  $Z_{T_+} = (f_1 \otimes f_3) \circ (1_{a_1^+ \otimes a_2^+ \otimes a_3^+} \otimes f_5 \otimes 1_{a_8^+})$ and $Z_{T_-} = f_2 \circ f_4$
- $Z_T := Z_{T_+} \otimes Z_{T_-} \in \mathcal{C}_+\left(t(\sigma_+), t(\tau_+)\right) \otimes \mathcal{C}_-\left(t(\sigma_-), t(\tau_-)\right)$
- Define category $\mathcal{NCP}$:
  - Objects in $\mathcal{NCP}$ are given by Σ
  - For $\sigma, \tau \in \Sigma$, the morphism space is defined by

$$\mathcal{NCP}(\sigma, \tau) := \mathbb{C}\text{-span}\left\{Z_T : T \in NCP(\sigma, \tau)\right\}$$

- $\mathcal{NCP}$ is a C*-tensor category and $\mathcal{C}_{\pm}$ sit inside $\mathcal{NCP}$ as full $*$-subcategories

Preliminaries- Free product of categories

- $\mathcal{C}_+ * \mathcal{C}_-$ is the "unitary projection completion" of $\mathcal{NCP}$, i.e.,
  - Obj $(\mathcal{C}_+ * \mathcal{C}_-) := \{(\sigma, p) \ : \ \sigma \in \Sigma$ and $p \in \mathcal{NCP}(\sigma, \sigma), \quad p^2 = p^* = p\}$
  - For $(\sigma, p), (\tau, q) \in$ Obj $(\mathcal{C}_+ * \mathcal{C}_-)$,

$$(\mathcal{C}_+ * \mathcal{C}_-)((\sigma, p), (\tau, q)) := q \circ \mathcal{NCP}(\sigma, \tau) \circ p$$

- $\mathcal{C}_+ * \mathcal{C}_-$ is a semi-simple C*-tensor category containing $\mathcal{C}_\pm$ as full subcategories
- Identify $\Sigma \ni \sigma \leftrightsquigarrow (\sigma, 1_\sigma) \in \text{Obj}(\mathcal{C}_+ * \mathcal{C}_-)$
- $\text{Irr}(\mathcal{C}_+ * \mathcal{C}_-)$ is given by words with alternating letters from $\text{Irr}(\mathcal{C}_+)$ and $\text{Irr}(\mathcal{C}_-)$, and the empty word

Annular algebra of free product of categories

- $\mathcal{C}, \mathcal{D}$ - rigid, semi-simple C*-tensor categories with simple unit objects
- $\mathbf{I}_{\mathcal{C}}$ (respectively $\mathbf{I}_{\mathcal{D}}$) be a set of representatives of the isomorphism classes of simple objects in $\mathcal{C}$ (respectively $\mathcal{D}$) excluding the isomorphism class of the unit object
- $\mathrm{Irr}(\mathcal{C} * \mathcal{D})$ is given by words (including the empty one) with letters coming alternatively from $\mathbf{I}_{\mathcal{C}}$ and $\mathbf{I}_{\mathcal{D}}$
- $\mathbf{W}$ be the subset of these words with **strictly positive and even** length, such that the first letter comes from $\mathbf{I}_{\mathcal{C}}$
- $\Lambda := \{\emptyset\} \cup \mathbf{I}_{\mathcal{C}} \cup \mathbf{I}_{\mathcal{D}} \cup \mathbf{W}$
- Λ is not full - the alternating words of odd length and the alternating words of even length starting with a letter from $\mathbf{I}_{\mathcal{D}}$ do not appear in Λ

Annular representation category of free product of categories

Lemma

$Rep(\mathcal{A}\Lambda)$ and the representation category of the tube algebra $\mathcal{A}$ of $\mathcal{C} * \mathcal{D}$, are unitarily equivalent as linear $*$-categories.

An independent fact

- $\mathcal{B}$ be an arbitrary rigid C^*-tensor category, and $\Gamma \subseteq \text{Obj } \mathcal{B}$ be an arbitrary weight set containing $\mathbb{1}$ which is "essentially full"
- $\mathcal{J}\Gamma_0 := \mathcal{A}\Gamma \cdot \mathcal{A}\Gamma_{\mathbb{1},\mathbb{1}} \cdot \mathcal{A}\Gamma$, the ideal in $\mathcal{A}\Gamma$ generated by $\mathcal{A}\Gamma_{\mathbb{1},\mathbb{1}}$
- $\Gamma = \text{Irr}(\mathcal{C})$, we write $\mathcal{J}\mathcal{C}_0$ for $\mathcal{J}\Gamma_0$
- $Rep_0(\mathcal{A}\Gamma)$ - category of **admissible representations of the fusion algebra** with respect to Γ.
  That is, $Rep_0(\mathcal{A}\Gamma) \cong Rep(C^*_u(\mathcal{B}))$
- $Rep_+(\mathcal{A}\Gamma) := Rep(\mathcal{A}\Gamma/\mathcal{J}\Gamma_0)$ is the representations of $\mathcal{A}\Gamma$ which contain $\mathcal{J}\Gamma_0$ in their kernel

Proposition

For any essentially full weight set Γ,

$$Rep(\mathcal{A}\Gamma) \cong Rep_0(\mathcal{A}\Gamma) \oplus Rep_+(\mathcal{A}\Gamma)$$

Back to our setup

- We may consider $\mathcal{A}\mathbf{W}$ as a $*$-subalgebra of $\mathcal{A}\Lambda$ (as $\mathbf{W} \subset \Lambda$)

Lemma

- *$\mathcal{A}\mathbf{W}$ is a direct summand of the algebra $\mathcal{A}\Lambda$*
- *As $*$-algebras, $\mathcal{A}\Lambda/\mathcal{J}\Lambda_0 \cong \mathcal{A}\mathcal{C}/\mathcal{J}\mathcal{C}_0 \oplus \mathcal{A}\mathcal{D}/\mathcal{J}\mathcal{D}_0 \oplus \mathcal{A}\mathbf{W}$.*

- $Rep(\mathcal{A}\Lambda) \cong Rep_0(\mathcal{A}\Lambda) \oplus Rep_+(\mathcal{A}\mathcal{C}) \oplus Rep_+(\mathcal{A}\mathcal{D}) \oplus Rep(\mathcal{A}\mathbf{W})$
- terms of our interest - $Rep_0(\mathcal{A}\Lambda)$ and $Rep(\mathcal{A}\mathbf{W})$

Annular representation category of free product of categories

Proposition

As $*$-algebras, $\mathcal{A}\mathbf{W} \cong \bigoplus_{[w]\in\mathbf{W}_0} M_{|w|}(\mathbb{C}) \otimes \mathbb{C}[\mathbb{Z}]$

Thus, $Rep(\mathcal{A}\mathbf{W}) \cong Rep(\mathbb{Z})^{\oplus \mathbf{W}_0}$, where $\mathbf{W}_0$- set of cyclic equivalence classes of words in $\mathbf{W}$

Proposition

$$C_u^*(\mathcal{C} * \mathcal{D}) \cong C_u^*(\mathcal{C}) * C_u^*(\mathcal{D})$$

Hence, $Rep_0(\mathcal{A}\Lambda) \cong Rep(C_u^*(\mathcal{C} * \mathcal{D})) \cong Rep(C_u^*(\mathcal{C}) * C_u^*(\mathcal{D}))$ as W-categories.*

Annular representation category of free product of categories

- Substituting these in the following,

$$Rep(\mathcal{A}\Lambda) \cong Rep_0(\mathcal{A}\Lambda) \oplus Rep_+(\mathcal{AC}) \oplus Rep_+(\mathcal{AD}) \oplus Rep(\mathcal{A}\mathbf{W})$$

we get end up with the main result:

Theorem

Let $\mathcal{C}$ and $\mathcal{D}$ be rigid C^*-tensor categories. Then as W^*-categories,

$$Rep(\mathcal{A}(\mathcal{C} * \mathcal{D})) \cong Rep(C_u^*(\mathcal{C}) * C_u^*(\mathcal{D})) \oplus Rep_+(\mathcal{AC}) \oplus Rep_+(\mathcal{AD}) \oplus Rep(\mathbb{Z})^{\oplus \mathbf{W}_0}$$

Application - Fuss-Catalan planar algebra

- $\mathcal{C}$ and $\mathcal{D}$ are **weakly Morita equivalent (wMe)** if there is a rigid C*-2 category with two 0-cells, say, 0 and 1, such that $\mathrm{End}(0) \cong \mathcal{C}$ and $\mathrm{End}(1) \cong \mathcal{D}$
  - $\mathcal{C}$ and $\mathcal{D}$ are wMe $\implies$ $Rep(\mathcal{A}\mathcal{C}) \cong Rep(\mathcal{A}\mathcal{D})$
- $N \subseteq M \leftrightsquigarrow P^{N \subseteq M}$, the "even parts" $\mathcal{C}_{NN}$ and $\mathcal{C}_{MM}$ are wMe
- $Rep(\mathcal{A}P^{N\subseteq M}) \cong Rep(\mathcal{A}\mathcal{C}_{NN}) \cong Rep(\mathcal{A}\mathcal{C}_{NN})$
- $\mathcal{TLJ}_0(\delta)$ be the even part of the Temperley-Lieb-Jones subfactor planar algebra $TLJ(\delta)$

Proposition

The even part of Fuss-Catalan planar algebra $FC(\alpha, \beta)$ is wMe to $\mathcal{TLJ}_0(\alpha) * \mathcal{TLJ}_0(\beta)$.

Thus, $Rep(\mathcal{A}FC(\alpha, \beta)) \cong Rep(\mathcal{A}\left(\mathcal{TLJ}_0(\alpha) * \mathcal{TLJ}_0(\beta)\right))$

- $\mathcal{A}\left(\mathcal{TLJ}_0(\delta)\right)$ is fully described by Jones-Reznikoff

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Thank You!