CHARACTERIZATION OF MAJORIZATION IN l^1

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Majorization

- Suppose $x, y \in \mathbb{R}^n$. Then x is said to be majorized by y if

$$\begin{aligned} \sum_{i=1}^{k} x_i^{\downarrow} &\leq \sum_{i=1}^{k} y_i^{\downarrow} \quad \text{for } 1 \leq k \leq n-1 \text{ and} \\ \sum_{i=1}^{n} x_i &= \sum_{i=1}^{n} y_i. \end{aligned}$$

- $x \preceq y$

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2. **Horn Theorem [5]** $x \preceq y$ if and only if $x = Qy$ for some orthostochastic matrix Q. A square matrix Q is said to be orthostochastic matrix if it is Schur-square of a orthogonal matrix i.e. $Q = (Q_{ij}) = (U_{ij}^2)$, where U is a orthogonal matrix.

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3. **Schur-Horn Theorem [5]** Given a self-adjoint $n \times n$ matrix H having eigenvalue list in y, there is a basis for which H has diagonal entries x if and only if $x \preceq y$.
4. $x \preceq y$ if and only if $\sum_{j=1}^{n} g(x_j) \leq \sum_{j=1}^{n} g(y_j)$ for any convex function g on $\mathbb{R}$ [4].

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3. *Arveson and Kadison[3] obtained other characterizations in using different methods, in a similar kind of settings.*
4. *More recently, Kaftal and Weiss [6] established infinite dimensional Schur-Horn theorem for sequences decreasing monotonically to zero.*

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Let $\xi = \{\xi_j\} \in l^1$, the positive part of the sequence ξ is $\xi^+ = (\xi_1^+, \xi_2^+, \dots)$
and
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Let $\xi^{+\downarrow} = (\xi_1^{+\downarrow}, \xi_2^{+\downarrow}, \dots)$ and $\xi^{-\downarrow} = (\xi_1^{-\downarrow}, \xi_2^{-\downarrow}, \dots)$, where
$\xi_1^{+\downarrow} \geq \xi_2^{+\downarrow} \geq \dots$ is the decreasing rearrangement of components of the sequence ξ^+ and
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$\xi_1^{-\downarrow} \geq \xi_2^{-\downarrow} \geq \dots$ is the decreasing rearrangement of components of the sequence ξ^-.
Without loss of generality, we redefine ξ^+ by $\xi^{+\downarrow}$ and ξ^- by $\xi^{-\downarrow}$.

Majorization in l^1

Definition

Let $\xi \ = \{\xi_j\}$ and $\eta \ = \{\eta_j\}$ be two non-negative sequences in l^1. We say that We say that ξ is weekly majorized by η if $\sup_\pi \sum_{j=1}^{k} \xi_{\pi(j)} \leq \sup_\pi \sum_{j=1}^{k} \eta_{\pi(j)}$ for $k \in \mathbb{N}$, where π is a permutation on $\mathbb{N}$. We denote it by $\xi \prec_w \eta$

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Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. One can think x as a sequence of l^1 by setting $x_k = 0$ for all $k > n$.
Let $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ be two elements in $\mathbb{R}^n$. Then $x \preceq y$ if and only if $x \prec y$.

Definition

Let $\eta \in l^1$. Then η is said to be pure if η^- and η^+ both either in c_{00} or not in c_{00}, where c_{00} denotes the space of all finite sequences.

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Theorem (A)

Let H be a self-adjoint operator on a separable Hilbert space K and $\xi = \{\xi_j\} \in l^1$. Suppose $\eta = \{\eta_j\} \in l^1$ is the eigenspectrum of H and pure. If $\xi \preceq \eta$, then there exists an orthonormal basis of K which is the union of $\{\phi_j\}_{j=1}^{\infty}$ and $\{f_j\}_{j=1}^{m}$ $(0 \leq m \leq \infty)$ such that $\langle H\phi_j, \phi_j \rangle = \xi_j$ for $j \in \mathbb{N}$ and $\langle Hf_j, f_j \rangle = 0$ for $j = 1, 2, 3, \ldots, m$.

Schur-Horn type theorem

Theorem (B)

Let K be a separable Hilbert space and $\xi, \eta \in l_1$. Suppose η is pure. If $\xi \preceq \eta$, then there exists an orthonormal basis of K which is the union of $\{\phi_j\}_{j=1}^{\infty}$ and $\{f_j\}_{j=1}^{m}$ $(0 \leq m \leq \infty)$ and a self-adjoint compact operator H on K such that $\{\eta_j : j \in \mathbb{N}\}$ is the eigenspectrum of H and $\langle H\phi_j, \phi_j\rangle = \xi_j$ for $j \in \mathbb{N}$, $\langle Hf_j, f_j\rangle = 0$ for $j = 1, 2, ..., m$.

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Proof: Let $\{\psi_j\}_{j=1}^{\infty}$ be any orthonormal basis of K. Let

$$H(x) = \sum_{j=1}^{\infty} \eta_j \langle x, \psi_j \rangle \psi_j \text{ for all } x \in K.$$

- H is bounded, self-adjoint and compact operator on K.
- $\{\eta_j : j \in \mathbb{N}\}$ is the eigenspectrum of H.

The proof of the theorem follows from Theorem(A).

Hardy-Littlewood-pólya type theorem

Let $\xi = \{\xi_j\} \in l^1$. Denote a new sequence $\widehat{\xi} := \{\widehat{\xi}_j\}$ by including finite or infinite number of zeros as components in the sequence ξ.

Theorem

Let $\xi = \{\xi_j\}, \eta = \{\eta_j\} \in l^1$ and η is pure. Then $\xi \preceq \eta$ iff $\widehat{\xi} = M\eta$ for some infinite matrix $M = (m_{ij})$, with $m_{ij} \geq 0$ and $\sum_{j=1}^{\infty} m_{ij} = 1, \sum_{i=1}^{\infty} m_{ij} = 1$ for $i,j \in \mathbb{N}$, where $\widehat{\xi}$ is defined above

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Proof: Assume that $\xi \preceq \eta$.
Let K be a separable Hilbert space with an orthonormal basis $\{\psi_j : j \in \mathbb{N}\}$.
Then there exists a self-adjoint, compact operator H and an orthonormal basis $\{\phi_j\}_{j=1}^{\infty} \cup \{f_j\}_{j=1}^{m}$ such that $\{\eta_j : j \in \mathbb{N}\}$ is the eigenspectrum of H and $\langle H\phi_j, \phi_j\rangle = \xi_j$ for $j \in \mathbb{N}$, $\langle Hf_j, f_j\rangle = 0$ for $j = 1, 2, \ldots, m$.

Proof cont...

- Denote $\{\phi_j^{'}\}_{j=1}^{\infty} = \{\phi_j\}_{j=1}^{\infty} \cup \{f_j\}_{j=1}^{m}$
- $U(\psi_j) = \phi_j^{'}$ for $j \in \mathbb{N}$.

Then
$$\widehat{\xi}_j = \langle H\phi_j^{'}, \phi_j^{'} \rangle = \langle H(U\psi_j), U\psi_j \rangle = \left\langle \sum_{k=1}^{\infty} \eta_k \langle U\psi_j, \psi_k \rangle \psi_k, U\psi_j \right\rangle.$$
Hence
$$\widehat{\xi}_j = \sum_{k=1}^{\infty} \eta_k \langle U\psi_j, \psi_k \rangle \langle \psi_k, U\psi_j \rangle = \sum_{k=1}^{\infty} \eta_k \mid \langle U\psi_j, \psi_k \rangle \mid^2 .$$
Set $m_{jk} = \mid \langle U\psi_j, \psi_k \rangle \mid^2$ for $j, k \in \mathbb{N}$. Then $m_{jk} \geq 0$ and $\widehat{\xi} = M\eta$, where $M = (m_{ij})$.

Proof cont...

Conversely, let $\widehat{\xi} = M\eta$.

$$
\begin{aligned}
\sum_{j=1}^{n} \widehat{\xi}_j^{\,+} &= \sum_{j=1}^{N} \widehat{\xi}_j^{\,+} = \sum_{j=1}^{N} \xi_j = \sum_{j=1}^{N}\sum_{k=1}^{\infty} m_{jk}\eta_k \\
&\leq \sum_{j=1}^{N}\sum_{k=1}^{\infty} m_{j2k}\eta_k^+ = \sum_{k=1}^{\infty}\sum_{j=1}^{N} m_{j2k}\eta_k^+ \\
&= \sum_{k=1}^{\infty} S_k\eta_k^+, \qquad \text{where } S_k = \sum_{j=1}^{N} m_{j2k} \\
&\leq \sum_{k=1}^{N-1} S_k\eta_k^+ + \sum_{k=N}^{\infty} S_k\eta_N^+ \\
&\leq \sum_{k=1}^{N-1} S_k\eta_k^+ + \left(N - \sum_{k=1}^{N-1} S_k\right)\eta_N^+ \leq \sum_{k=1}^{n} \eta_k^+.
\end{aligned}
$$

So $\widehat{\xi}^+ \prec_w \eta^+$. Similarly $\widehat{\xi}^- \prec_w \eta^-$. Also $\sum_{j=1}^{\infty} \widehat{\xi}_j = \sum_{k=1}^{\infty} \eta_k$. Hence $\widehat{\xi} \preceq \eta$.

Relations between majorization in l^1 and convex function.

Theorem

Let $\xi = \{\xi_j\}, \eta = \{\eta_j\} \in l^1$ and η is pure. Assume that $\xi \preceq \eta$. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a continuous convex function. Then the following hold.

1. If $\{g(\eta_j)\} \in l_1$, then $\sum_{j=1}^{\infty} g(\xi_j) \leq \sum_{j=1}^{\infty} g(\eta_j)$.

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1. If $\{g(\eta_j)\} \in l_1$, then $\sum_{j=1}^{\infty} g(\xi_j) \leq \sum_{j=1}^{\infty} g(\eta_j)$.
2. If $g(\eta_j)$'s has same sign except finitely many, then $\sum_{j=1}^{\infty} g(\widehat{\xi}_j) \leq \sum_{j=1}^{\infty} g(\eta_j)$, where $\widehat{\xi} = \{\widehat{\xi}_j\}$ defined above.

Characterization of majorization in l^1

Theorem

Let $\alpha = \{\alpha_j\}, \beta = \{\beta_j\} \in l^1$. Then the following are equivalent

1. $\alpha \preceq \beta$
2. $\sum_{j=1}^{\infty}(\alpha_j - t)^+ \leq \sum_{j=1}^{\infty}(\beta_j - t)^+, \ \sum_{j=1}^{\infty}(t - \alpha_j)^+ \leq \sum_{j=1}^{\infty}(t - \beta_j)^+$ for all $t \in \mathbb{R}$ and $\sum_{j=1}^{\infty} \alpha_j = \sum_{j=1}^{\infty} \beta_j$.

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1. $\alpha \preceq \beta$
2. $\sum_{j=1}^{\infty}(\alpha_j - t)^+ \leq \sum_{j=1}^{\infty}(\beta_j - t)^+, \quad \sum_{j=1}^{\infty}(t-\alpha_j)^+ \leq \sum_{j=1}^{\infty}(t-\beta_j)^+$ for all $t \in \mathbb{R}$ and $\sum_{j=1}^{\infty}\alpha_j = \sum_{j=1}^{\infty}\beta_j$.

Corollary

Let $\alpha, \beta \in l^1$ with $\sum_{j=1}^{\infty}\alpha_j = \sum_{j=1}^{\infty}\beta_j$. If $\sum_{j=1}^{\infty}g(\alpha_j) \leq \sum_{j=1}^{\infty}g(\beta_j)$ for any convex function g on $\mathbb{R}$, then $\alpha \preceq \beta$.

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Thank you