Choquet theory on state spaces of C*-algebras and the hyperrigidity conjecture

Raphaël Clouâtre

University of Manitoba

Recent Advances in Operator Theory and Operator Algebras
Indian Statistical Institute, Bangalore
December 13, 2018

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Example (C*-covers of the disc algebra)

Let $A(\mathbb{D})$ be the disc algebra. Consider unital completely isometric maps

$$\varphi_1 : A(\mathbb{D}) \to \mathrm{C}(\overline{\mathbb{D}}), \quad \varphi_2 : A(\mathbb{D}) \to \mathrm{C}(\mathbb{T}), \quad \varphi_3 : A(\mathbb{D}) \to \mathfrak{T}$$

defined as

$$\varphi_1(f) = f, \quad \varphi_2(f) = f|_{\mathbb{T}}, \quad \varphi_3(f) = M_f$$

for every $f \in A(\mathbb{D})$. Then,

$$(\mathrm{C}(\overline{\mathbb{D}}), \varphi_1), \quad (\mathrm{C}(\mathbb{T}), \varphi_2), \quad (\mathfrak{T}, \varphi_3)$$

are C*-covers of $A(\mathbb{D})$.

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Yes, $\mathcal{M}$ has a C*-envelope.

Theorem (Hamana 1979)

There is a C*-*cover* $(\mathrm{C}^*_e(\mathcal{M}), \varepsilon)$ *of* $\mathcal{M}$ *with the property that given any* C*-*cover* $(\mathfrak{A}, \varphi)$ *of* $\mathcal{M}$, *there is a unital* $*$-*representation* $\pi : \mathfrak{A} \to \mathrm{C}^*_e(\mathcal{M})$ *such that* $\pi \circ \varphi = \varepsilon$.

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How can we identify the C*-envelope?

Inspiration from uniform algebra theory: the Shilov boundary

X compact metric space, $\mathcal{A} \subset \mathrm{C}(X)$ uniform algebra
A closed subset $\Delta \subset X$ is a boundary for $\mathcal{A}$ if

$$\max_{x \in X} |\varphi(x)| = \max_{x \in \Delta} |\varphi(x)|, \quad \varphi \in \mathcal{A}.$$

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Definition

The Shilov boundary of $\mathcal{A}$ is the smallest boundary $\Sigma_{\mathcal{A}} \subset X$ for $\mathcal{A}$.

For every boundary $\Delta \subset X$, the surjective restriction map $\mathrm{C}(\Delta) \to \mathrm{C}(\Sigma_{\mathcal{A}})$ is (completely) isometric on $\mathcal{A}$.

Peak points and the Choquet boundary

Theorem

Let $\xi \in X$. Then, the following statements are equivalent.

- *The point ξ is a peak point for $\mathcal{A}$: there is $\varphi \in \mathcal{A}$ with the property that*

$$\varphi(\xi) = 1 > |\varphi(x)|, \quad x \neq \xi.$$

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Furthermore,

$$\overline{\textit{Choquet boundary}} = \Sigma_{\mathcal{A}}.$$

Arveson's non-commutative uniform algebra theory

Definition (Arveson 1969)

Let $\mathcal{M}$ be a concretely represented unital operator space.

- A unital completely contractive linear map $\varphi : \mathrm{C}^*(\mathcal{M}) \to B(\mathfrak{H})$ is said to have the unique extension property with respect to $\mathcal{M}$ if it is the unique unital completely contractive extension to $\mathrm{C}^*(\mathcal{M})$ of $\varphi|_{\mathcal{M}}$.
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Theorem (Arveson 1969)

Let $\mathcal{F}$ be a set of unital $*$-representations of $\mathrm{C}^*(\mathcal{M})$ which have the unique extension property with respect to $\mathcal{M}$. Assume that $\varepsilon = \bigoplus_{\pi\in\mathcal{F}} \pi$ is completely isometric on $\mathcal{M}$. Then $(\varepsilon(\mathrm{C}^*(\mathcal{M})), \varepsilon)$ is the C^*-envelope of $\mathcal{M}$.

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Such a set $\mathcal{F}$ always exists (Muhly-Solel 1998, Dritschel–McCullough 2005).

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Such a set $\mathcal{F}$ always exists (Muhly-Solel 1998, Dritschel–McCullough 2005). The $*$-representations can even be chosen to be irreducible (Arveson 2008, Davidson–Kennedy 2015).

Hyperrigidity and the unique extension property

Definition

A concretely represented unital operator space $\mathcal{M}$ is said to be hyperrigid if every unital $*$-representation of $C^*(\mathcal{M})$ has the unique extension property with respect to $\mathcal{M}$.

Note that this notion depends on the choice of representation of $\mathcal{M}$. However, if $\mathcal{M}$ is known to be hyperrigid in some representation, then it will be automatically be hyperrigid inside of its C^*-envelope.

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Theorem (Kennedy–Shalit 2015)

The Arveson-Douglas essential normality conjecture can be rephrased in terms of hyperrigidity of a natural unital operator space.

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Theorem (Arveson 2011)

A concretely represented unital operator space $\mathcal{M}$ is hyperrigid if and only if for every unital $*$-representation $\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathfrak{H})$ and every sequence of unital completely positive maps

$$\varphi_n : \pi(\mathrm{C}^*(\mathcal{M})) \to B(\mathfrak{H}), \quad n \in \mathbb{N}$$

satisfying

$$\lim_{n\to\infty} \|\varphi_n(\pi(a)) - \pi(a)\| = 0, \quad a \in \mathcal{M},$$

we must have

$$\lim_{n\to\infty} \|\varphi_n(\pi(t)) - \pi(t)\| = 0, \quad t \in \mathrm{C}^*(\mathcal{M}).$$

Hyperrigidity in approximation theory

Theorem (Korovkin 1953)

For each $n \in \mathbb{N}$*, let* $\varphi_n : \mathrm{C}[0,1] \to \mathrm{C}[0,1]$ *be a (completely) positive linear map and assume that*

$$\lim_{n\to\infty} \|\varphi_n(a) - a\| = 0$$

for every $a \in \{1, x, x^2\}$*. Then, it must be the case that*

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In order for a general unital operator space to be hyperrigid, is it sufficient for the non-commutative Choquet boundary to be maximal?

The conjecture and some supporting evidence

Hyperrigidity conjecture (Arveson 2011)

Let $\mathcal{M}$ be a concretely represented unital operator space. Then, $\mathcal{M}$ is hyperrigid if and only if every irreducible $*$-representation of $\mathrm{C}^*(\mathcal{M})$ has the unique extension property with respect to $\mathcal{M}$.

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Some examples of unital operator spaces satisfying the hyperrigidity conjecture:

- multiplier algebras of certain reproducing kernel Hilbert spaces (C.–Hartz 2017)
- tensor algebras of certain directed graphs. (Dor On–Salomon 2018)

Irreducible $*$-representations as building blocks

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Lemma (Arveson 2011)

Let $\mathcal{M}$ be a concretely represented unital operator space. For each $\lambda \in \Lambda$, let $\pi_\lambda : \mathrm{C}^*(\mathcal{M}) \to B(\mathfrak{H}_\lambda)$ be a unital $*$-representation. Then,

$$\bigoplus_{\lambda\in\Lambda} \pi_\lambda : \mathrm{C}^*(\mathcal{M}) \to \bigoplus_{\lambda\in\Lambda} B(\mathfrak{H}_\lambda)$$

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Recall that the spectrum of a C^*-algebra is the set of unitary equivalence classes of its irreducible representations.

Theorem (Arveson 2011)

Let $\mathcal{M}$ be a concretely represented unital operator space such that $\mathrm{C}^*(\mathcal{M})$ has countable spectrum. Then, $\mathcal{M}$ satisfies the hyperrigidity conjecture.

Linearizing the problem

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Lemma

The following statements are equivalent.

(i) *We have $\pi = \Pi$.*

(ii) *There is a family of states on $B(\mathfrak{H})$ which separate $(\Pi - \pi)(C^*(\mathcal{M}))$ and restrict to pure states on $\pi(C^*(\mathcal{M}))$.*

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How do we manufacture such a family of states?

Unperforated pairs

Definition

Let $\mathfrak{A}$ be a unital C*-algebra. Let $\mathcal{S}$ and $\mathcal{T}$ be self-adjoint subspaces of $\mathfrak{A}$. We say that the pair $(\mathcal{S}, \mathcal{T})$ is unperforated if for every pair of self-adjoint elements $a \in \mathcal{S}, b \in \mathcal{T}$ such that $a \leq b$, we can find another self-adjoint element $b' \in \mathcal{T}$ with the property that $\|b'\| \leq \|a\|$ and $a \leq b' \leq b$.

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Example

If $\mathfrak{B} \subset \mathfrak{A}$ is a unital C*-subalgebra that commutes with a self-adjoint subspace $\mathcal{S} \subset \mathfrak{A}$, then the pair $(\mathcal{S}, \mathfrak{B})$ is unperforated.

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- Even in finite-dimensional settings, unperforated pairs appear elusive in the absence of some form of commutativity.
- Let $\mathfrak{A}$ be a unital C*-algebra and let $\mathfrak{B} \subset \mathfrak{A}$ be a unital C*-subalgebra with the weak expectation property. Then, the pair $(\mathfrak{A}, \mathfrak{B})$ is "approximately" unperforated. (C. 2018)

A local version of the conjecture?

X be a compact metric space
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Example

Let X be a compact metric space and $\mathcal{A} \subset \mathrm{C}(X)$ be a uniform algebra. Let $\xi \in X$ be a peak point for $\mathcal{A}$, so that there is a function $\varphi \in \mathcal{A}$ with the property that

$$|\varphi(y)| < \varphi(\xi) = 1$$

for each $y \in X, y \neq \xi$. Then, $\lim_{n \to \infty} \|\varphi^n f\| = |f(\xi)|$ for every $f \in \mathrm{C}(X)$.

Characteristic sequences

Definition

Let $\mathfrak{A}$ be a unital C*-algebra and let ψ be a state on $\mathfrak{A}$. A sequence $(\Delta_n)_n$ in $\mathfrak{A}$ is said to be a *characteristic sequence* for ψ if the following conditions are satisfied:

(a) $\|\Delta_n\| = 1$ for every $n \in \mathbb{N}$,

(b) $\lim_{n\to\infty} \psi(\Delta_n) = 1$, and

(c) $\limsup_{n\to\infty} \|\Delta_n^* a \Delta_n\| \leq |\psi(a)|$ for every $a \in \mathfrak{A}$.

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Example

Let $\mathcal{A}_d$ denote the norm closure of the polynomial multipliers on the Drury-Arveson space. Let $\mathfrak{T}_d = \mathrm{C}^*(\mathcal{A}_d)$ denote the Toeplitz algebra. Every pure state on $\mathfrak{T}_d$ admits a characteristic sequence in $\mathfrak{K} + \mathcal{A}_d$.

Local hyperrigidity in general

$\mathcal{M}$ concretely represented unital operator space
$\pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathfrak{H})$ unital $*$-representation
$\Pi : \mathrm{C}^*(\mathcal{M}) \to B(\mathfrak{H})$ unital completely contractive map such that

$$\pi(a) = \Pi(a), \quad a \in \mathcal{M}$$

Theorem (C. 2018)

Let ψ be a state on $\mathrm{C}^*(\mathcal{M})$ which admits a characteristic sequence $(\Delta_n)_n$ in $\mathcal{M}$. Then, we have

$$\lim_{n\to\infty} \|\pi(\Delta_n)^*(\Pi(t) - \pi(t))\pi(\Delta_n)\| = 0$$

for every $t \in \mathrm{C}^*(\mathcal{M})$.

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Note that although the conclusion is merely "local", so is the assumption!

Thank you!