On the commutants of weighted shifts on directed trees

Marek Ptak

University of Agriculture, Kraków

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Based on a joint work with Artur Płaneta and Piotr Dymek

Unilateral shift

Let $S\colon \ell^2 \to \ell^2$ be a unilateral shift given by

$$S(a_0, a_1, \dots) = (0, a_0, a_1, \dots).$$

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The ℓ^2 space and the Hardy space H^2 are unitary equivalent.

Definition

Let $H^\infty = L^\infty \cap H^2$ and let $\phi \in H^\infty$. Define $M_\phi\colon H^2 \to H^2$ by

$$(M_\phi f)(z) = \phi(z) f(z), \quad z \in \mathbb{D}.$$

Theorem (Brown, Halmos)

Operator S is unitary equivalent to the multiplication operator M_z and commutant of M_z is equal to the space $\{M_\phi\colon \phi \in H^\infty\}$.

Weighted shift

Let $\boldsymbol{\lambda} = \{\lambda_n\}_{n=0}^{\infty} \subset (0, +\infty)$ and let $S_{\boldsymbol{\lambda}}$ be a bounded weighted shift given by

$$S_{\boldsymbol{\lambda}}(a_0, a_1, \ldots,) = (0, \lambda_0 a_0, \lambda_1 a_1, \ldots).$$

Definition

Let $\boldsymbol{\beta} = \{\beta(n)\}_{n=0}^{\infty} \subset (0, +\infty)$ such that $\beta(0) = 1$ and let $H^2(\boldsymbol{\beta})$ be a weighted Hardy space i.e,

$$H^2(\boldsymbol{\beta}) = \left\{ f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \colon \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \beta(n) < \infty \right\}.$$

Weighted shift cd.

Definition

- For a formal power series

$$\phi(z)=\sum_{n=0}^{\infty}\hat{\phi}(n)z^n$$

and a function

$$f(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n\in H^2(\beta)$$

we define its multiplication by

$$(\phi f)(z)=\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\hat{\phi}(k)\hat{f}(n-k)\Big)z^n=\sum_{n=0}^{\infty}(\hat{\phi}*\hat{f})(n)z^n.$$

Weighted shift cd.

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we define its multiplication by
$$(\phi f)(z) = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \hat{\phi}(k) \hat{f}(n-k) \Big) z^n = \sum_{n=0}^{\infty} (\hat{\phi} * \hat{f})(n) z^n.$$
- Let $H^\infty(\beta) = \{\phi(z) = \sum_{n=0}^{\infty} \hat{\phi}(n) z^n \colon \phi H^2(\beta) \subset H^2(\beta)\}$.

Weighted shift

Definition

For $\phi \in H^\infty(\beta)$, $f \in H^2(\beta)$ we define multiplication operator by

$$(M_\phi f)(z) = (\phi f)(z).$$

Theorem (Gellar, Kelley, Shields)

Weighted shift $S_{\boldsymbol{\lambda}}$ is unitary equivalent with the multiplication operator $M_z \colon H^2(\beta) \to H^2(\beta)$, with

$$\beta(n) = \lambda_0 \cdot \ldots \cdot \lambda_{n-1}.$$

For any $\phi \in H^\infty(\beta)$ operator M_ϕ is bounded and

$$\{M_z\}' = \{M_\phi \colon \phi \in H^\infty(\beta)\}.$$

Shift of arbitrary multiplicity

For a Hilbert space K let $\ell^2(K) = \{\{x_n\}_{n=0}^{\infty} \colon \sum_{n=0}^{\infty} \|x_n\|_K^2 < \infty\}$ and let $\boldsymbol{S}_K$ be a shift on $\ell^2(K)$ i.e.,

$$\boldsymbol{S}_K(x_0, x_1, \ldots) = (0, x_0, x_1, \ldots)$$

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Function space model of $\boldsymbol{S}_K$:

$$L^2(K) = \left\{f \colon \mathbb{T} \to K \text{ measurable } : \int_{\mathbb{T}} \|f(z)\|^2 \mathrm{d}m\right\} < \infty\}.$$

Any $f \in L^2(K)$ can be represented as

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n, \quad \hat{f}(n) \in K.$$

Define Hilbert Hardy space as

$$H^2(K) = \left\{f \in L^2(K) \colon f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n\right\}.$$

Shift of arbitrary multiplicity

Function space model of $\boldsymbol{S}_K$ cd.:

Let $F\colon \mathbb{T} \to \mathbf{B}(K)$ be measureable and essentially bounded. Define multiplication by F as the operator $M_F\colon L^2(K) \to L^2(K)$

$$(M_F f)(z) = F(z)f(z), \quad z \in \mathbb{T}.$$

Theorem (Lax, Halmos, Helson, Lowdenslager)

Operator $\boldsymbol{S}_K$ is unitary equivalent to the multiplication operator M_z and

$$\{M_z\}' = \{M_F|_{H^2(K)}\colon M_F H^2(K) \subset H^2(K)\}.$$

Graphs and Directed trees

A pair $\mathcal{G} = (V, \vec{E})$ is called a directed graph if V is a nonempty set and $\vec{E}$ is a subset of a set $(V \times V) \setminus \{(v, v) \colon v \in V\}$ (set V may be infinite).

Graphs and Directed trees

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During the talk we will use the following notation:

- $\mathrm{par}(u)$ – the parent of a vertex u (unique if exists),
- $\mathrm{Chi}(u)$ – the set of children of a vertex u,
- root – a vertex which is not a child of any vertex,
- V° – the set of vertices excluding roots,
- $\mathrm{par}^n(u)$ – the n-th parent of a vertex u (unique if exists),
- $\mathrm{Chi}^{\langle n \rangle}(W)$ – the set of descendants of the n-th order of a set W.

Graphs and Directed trees

Definition

A directed graph $\mathscr{T}$ is called a directed tree if the following conditions are satisfied:

- there are no circuits in $\mathscr{T}$,
- $\mathscr{T}$ is connected,
- every vertex which is not a root has a parent.

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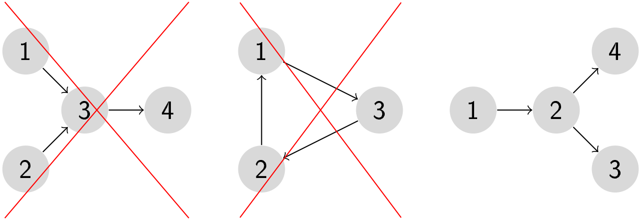

$\ell^2(V)$ space

Let $\mathscr{T} = (V, \vec{E})$ be a directed tree. We consider the Hilbert space $\ell^2(V)$, which consists of functions $f \colon V \to \mathbb{C}$ such that

$$\sum_{u \in V} |f(u)|^2 < \infty.$$

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$$\langle f, g \rangle := \sum_{u \in V} f(u)\overline{g(u)}.$$

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For every vertex u we define the function $e_u \in \ell^2(V)$ by formula

$$e_u(v) := \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \neq v \end{cases}.$$

Weighted shift on a directed tree

[JJS] Z. J. Jabłoński, I. B. Jung, J. Stochel, *Weighted shifts on directed trees*, Mem. Amer. Math. Soc. **216**, no. 1017, viii+107pp (2012).

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Let $\mathscr{T} = (V, \vec{E})$ be a directed tree and let $\boldsymbol{\lambda}\colon V^\circ \ni v \mapsto \lambda_v \in \mathbb{C}$. Assume that $\sup_{u\in V} \sum_{v\in \mathsf{Chi}(u)} |\lambda_v|^2 < \infty$.

For a function $f \in \ell^2(V)$ we define

$$(S_{\boldsymbol{\lambda}} f)(v) := \begin{cases} \lambda_v f(\mathsf{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{if } v \text{ is a root.} \end{cases}$$

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$S_{\boldsymbol{\lambda}} \in \mathbf{B}(\ell^2(V))$, is called a weighted shift on a directed tree.

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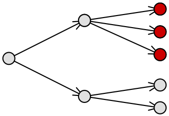
$$S_{\boldsymbol{\lambda}}(e_u) = \sum_{v\in\mathrm{Chi}(u)} \lambda_v e_v.$$



Weighted shift on a directed tree

Classical unilateral weighted shift

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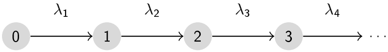

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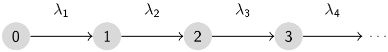

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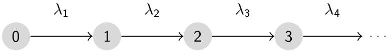

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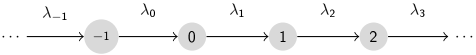

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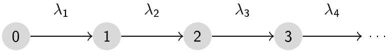

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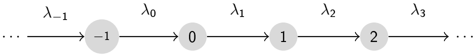

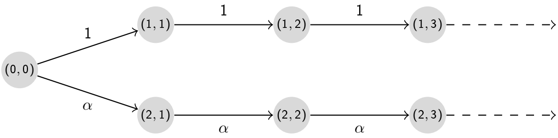

Example

Let $\mathscr{T}_2 = (V_2, E_2)$, $\alpha \in (0,1)$

$$V_2 = \big\{(0,0)\big\} \cup \big\{(i,j)\colon\, i \in \{1,2\},\ j \in \mathbb{N}\big\},$$
$$E_2 = \Big\{\big((0,0),(i,1)\big)\colon\, i \in \{1,2\}\Big\} \cup \Big\{\big((i,j),(i,j+1)\big)\colon\, i \in \{1,2\},\ j \in \mathbb{N}\Big\}.$$

Let S_λ be a weighted shift on $\mathscr{T}_2$ with weights

$$\lambda_{(i,j)} = \begin{cases} 1 & \text{for } i = 1 \text{ and } j \in \mathbb{N}, \\ \alpha & \text{for } i = 2 \text{ and } j \in \mathbb{N}. \end{cases}$$

Problem

Describe commutant of $S_{\boldsymbol{\lambda}}$, i.e $\{S_{\boldsymbol{\lambda}}\}' \ =?$.

Left-invertible analytic operator

We say that $T\colon \mathcal{H} \to \mathcal{H}$ is left-invertible if there exists an operator $L \in \mathbf{B}(\mathcal{H})$ such that $LT = I_{\mathcal{H}}$.

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$T \in \mathbf{B}(\mathcal{H})$ is left-invertible iff there exists a constant $c > 0$ such that

$$\|Tx\| \geq c\|x\| \text{ for all } x \in \mathcal{H}.$$

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Let $T \in \mathbf{B}(\mathcal{H})$. Operator T is said to be analytic if

$$\bigcap_{n=1}^{\infty} T^n(\mathcal{H}) = \{0\}.$$

Examples of left-invertible analytic operators

- Unilateral shift
- Weighted shifts on leafless and rooted directed trees which satisfy

$$\inf_{u \in V} \sum_{v \in \mathrm{Chi}(u)} |\lambda_v|^2 > 0$$

(A directed tree $\mathscr{T} = (V, E)$ is rooted iff it has a root;
is leafless iff $\mathrm{Chi}(v) \neq \emptyset$ for all $v \in V$.)

- Shifts on generalized Dirichlet spaces
- Shifts on weighted Bergmann space with logarithmically subharmonic weights on the unit disc in the complex plane

Model of left-invertible analytic operator

For $T \in \mathbf{B}(\mathcal{H})$ which is left-invertible and analytic we define operators

$$T' := T(T^*T)^{-1} \text{ and } L := (T')^*.$$

Then

$$LT = I_{\mathcal{H}}.$$

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Shimourin's Model
Denote by $\mathcal{A}(\mathcal{N}(T^*))$ the set of all $\mathcal{N}(T^*)$-valued analytic functions on $\mathbb{D}_r$, where $r := \frac{1}{r(L)}$.

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$\boldsymbol{f} \in \mathcal{A}(\mathcal{N}(T^*))$ if there is a sequence $\hat{\boldsymbol{f}} = \{\hat{\boldsymbol{f}}(n)\}_{n=0}^{\infty} \subset \mathcal{N}(T^*)$ such that

$$\boldsymbol{f}(z) = \sum_{n=0}^{\infty} \hat{\boldsymbol{f}}(n) z^n, \qquad \text{for every } z \in \mathbb{D}_r.$$

Model of left-invertible analytic operator

Let us define a transformation $U\colon \mathcal{H} \to \mathcal{A}(\mathcal{N}(T^*))$ by the formula

$$(Uf)(z) = \sum_{n=0}^{\infty} (P_{\mathcal{N}(T^*)} L^n f) z^n, \quad z \in \mathbb{D}_r.$$

is well-defined and analytic in $\mathbb{D}_r$ for every $f \in \mathcal{H}$.

- Transformation U is injective.
- We define a new scalar product on the set $U(\mathcal{H})$ such that U is an isometry.
- Let $\mathscr{H}$ denote the set $U(\mathcal{H})$. It is a Hilbert space of $\mathcal{N}(T^*)$-valued analytic functions.

Model of left-invertible analytic operator

- T is unitary equivalent to the operator $\mathcal{T}$ of multiplication by z on $\mathcal{H}$.
- L is unitary equivalent to the operator $\mathcal{L} \in \mathbf{B}(\mathcal{H})$ given by the formula

$$\mathcal{L}(\boldsymbol{f})(z) = \frac{\boldsymbol{f}(z) - \boldsymbol{f}(0)}{z}.$$

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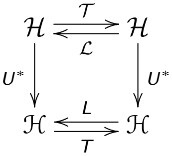

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$$\begin{array}{ccc} \mathcal{H} & \overset{\mathcal{T}}{\underset{\mathcal{L}}{\rightleftarrows}} & \mathcal{H} \\ {\scriptstyle U^*}\downarrow & & \downarrow{\scriptstyle U^*} \\ \mathscr{H} & \overset{L}{\underset{T}{\leftrightarrows}} & \mathscr{H} \end{array}$$

Every $\boldsymbol{f} \in \mathcal{H}$ can be represented as follows

$$\boldsymbol{f}(z) = \sum_{n=0}^{\infty} \hat{\boldsymbol{f}}(n) z^n,$$

where $\hat{\boldsymbol{f}}(n) := P_{\mathcal{N}(T^*)} L^n U^* \boldsymbol{f}$ for $n \in \mathbb{N}_0$.

Cauchy-type multiplication

We define $*\colon \mathbf{B}(\mathcal{N}(T^*))^{\mathbb{N}_0} \times \mathcal{N}(T^*)^{\mathbb{N}_0} \to \mathcal{N}(T^*)^{\mathbb{N}_0}$ given by

$$(\hat{\varphi} * \hat{f})(n) = \sum_{k=0}^{n} \hat{\varphi}(k)\hat{f}(n-k), \quad \hat{\varphi} \in \mathbf{B}(\mathcal{N}(T^*))^{\mathbb{N}_0},\ \hat{f} \in \mathcal{N}(T^*)^{\mathbb{N}_0}.$$

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$$\mathcal{D}(M_{\hat{\varphi}}) = \left\{\boldsymbol{f} \in \mathcal{H}\colon \text{ there is } \boldsymbol{g} \in \mathcal{H} \text{ such that } \hat{\varphi} * \hat{\boldsymbol{f}} = \hat{\boldsymbol{g}}\right\}.$$

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For every $\boldsymbol{f} \in \mathcal{D}(M_{\hat{\varphi}})$ there exists exactly one $\boldsymbol{g} \in \mathcal{H}$ satisfying equality $\hat{\varphi} * \hat{\boldsymbol{f}} = \hat{\boldsymbol{g}}$. In this situation, we set

$$M_{\hat{\varphi}}\boldsymbol{f} = \boldsymbol{g}, \quad \boldsymbol{f} \in \mathcal{D}(M_{\hat{\varphi}}).$$

We call $\hat{\varphi}\colon \mathbb{N}_0 \to \mathbf{B}(\mathcal{N}(T^*))$ the symbol of $M_{\hat{\varphi}}$.

Generalized multipliers

Lemma

For every $\boldsymbol{f} \in \mathcal{D}(M_{\hat{\varphi}})$ and $n \in \mathbb{N}_0$, we have the equality

$$\widehat{M_{\hat{\varphi}}\boldsymbol{f}}(n) = \sum_{k=0}^{n} \hat{\varphi}(k)\hat{\boldsymbol{f}}(n-k).$$

Generalized multipliers

Lemma

For every $\boldsymbol{f} \in \mathcal{D}(M_{\hat\varphi})$ and $n \in \mathbb{N}_0$, we have the equality

$$\widehat{M_{\hat\varphi}\boldsymbol{f}}(n) = \sum_{k=0}^{n} \hat\varphi(k)\hat{\boldsymbol{f}}(n-k).$$

If $\mathcal{D}(M_{\hat\varphi}) = \mathcal{H}$, then $M_{\hat\varphi} \in \mathbf{B}(\mathcal{H})$. If so, we call

- $\hat\varphi$ a generalized multiplier of T,
- $M_{\hat\varphi}$ a generalized multiplication operator by $\hat\varphi$,
- $\mathcal{GM}(T)$ the set of all generalized multipliers of the operator T,
- $\mathcal{M}(T)$ is the linear subspace of $\mathcal{GM}(T)$ consisting of all generalized multipliers whose all coefficients are scalar multiples of $I_{\mathcal{N}(T^*)}$.

Cauchy-type multiplication on $\mathcal{GM}(T)$

$\mathcal{GM}(T)$ is a linear subspace of $\mathbf{B}(\mathcal{N}(T^*))^{\mathbb{N}_0}$ and the function $\|\cdot\|\colon \mathcal{GM}(T) \to [0,\infty)$ given by the formula

$$\|\hat{\varphi}\| := \|M_{\hat{\varphi}}\|, \quad \hat{\varphi} \in \mathcal{GM}(T)$$

is a norm on $\mathcal{GM}(T)$.

Cauchy-type multiplication on $\mathcal{GM}(T)$

$\mathcal{GM}(T)$ is a linear subspace of $\mathbf{B}(\mathcal{N}(T^*))^{\mathbb{N}_0}$ and the function $\|\cdot\|\colon \mathcal{GM}(T) \to [0,\infty)$ given by the formula

$$\|\hat{\varphi}\| := \|M_{\hat{\varphi}}\|, \quad \hat{\varphi} \in \mathcal{GM}(T)$$

is a norm on $\mathcal{GM}(T)$.

We endow space $\mathcal{GM}(T)$ with the Cauchy-type multiplication

$$*\colon \mathbf{B}(\mathcal{N}(T^*))^{\mathbb{N}_0} \times \mathbf{B}(\mathcal{N}(T^*))^{\mathbb{N}_0} \to \mathbf{B}(\mathcal{N}(T^*))^{\mathbb{N}_0}$$

given by

$$(\hat{\varphi} * \hat{\psi})(k) = \sum_{j=0}^{k} \hat{\varphi}(j)\hat{\psi}(k-j), \quad \hat{\varphi}, \hat{\psi} \in \mathbf{B}(\mathcal{N}(T^*))^{\mathbb{N}_0}.$$

Banach algebra structure

Theorem

Let $T \in \mathbf{B}(\mathcal{H})$ be left-invertible and analytic. Then

(i) For every $n \in \mathbb{N}_0$, the sequence $\chi_{\{n\}} \mathrm{I}_{\mathcal{N}(T^*)}$ is a generalized multiplier and $\mathcal{T}^n = M_{\chi_{\{n\}} \mathrm{I}_{\mathcal{N}(T^*)}}$.

(ii) If $\hat{\varphi} \in \mathcal{GM}(T)$, then $M_{\hat{\varphi}}$ commutes with $\mathcal{T}$.

(iii) For all $\hat{\varphi}, \hat{\psi} \in \mathcal{GM}(T)$, the function $\hat{\varphi} * \hat{\psi}$ belongs to $\mathcal{GM}(T)$ and

$$M_{\hat{\varphi}} M_{\hat{\psi}} = M_{\hat{\varphi} * \hat{\psi}}.$$

(iv) The spaces $\mathcal{GM}(T)$, $\mathcal{M}(T)$ endowed with the Cauchy-type multiplication are the Banach algebras with a unit $\chi_{\{0\}} \mathrm{I}_{\mathcal{N}(T^*)}$. $\mathcal{M}(T)$ is commutative.

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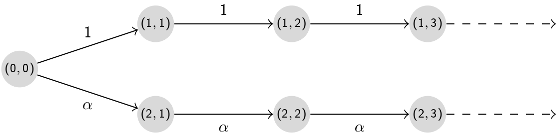
Remark

$\hat{\varphi} \colon \mathbb{N}_0 \to \mathbb{C} \operatorname{I}_{\mathcal{N}(T^*)} \subset \mathbf{B}(\mathcal{N}(T^*))$ and $\operatorname{supp} \hat{\varphi}$ is finite then $\hat{\varphi} \in \mathcal{M}(T)$,
$\hat{\varphi} \colon \mathbb{N}_0 \to \mathbf{B}(\mathcal{N}(T^*))$ and $\operatorname{supp} \hat{\varphi}$ is finite $\not\Rightarrow$ $\hat{\varphi} \in \mathcal{GM}(T)$,

Examples of General multipliers



Let $\mathscr{T}_2 = (V_2, E_2)$, $\alpha \in (0,1)$
$S_{\boldsymbol{\lambda}} \in \mathbf{B}(\ell^2(V_2))$ left-invertible and analytic,
$\{e_{00}, \alpha e_{11} - e_{21}\}$ is a basis of $\mathcal{N}(S^*_{\boldsymbol{\lambda}})$,

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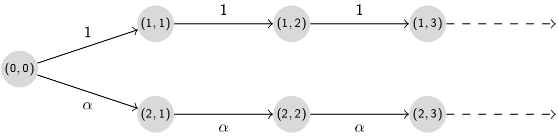

Let $\mathscr{T}_2 = (V_2, E_2)$, $\alpha \in (0, 1)$
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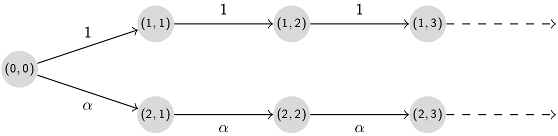
Example

$\hat\varphi\colon \mathbb{N}_0 \to \mathbf{B}(\mathcal{N}(S^*_{\boldsymbol\lambda}))$, $\hat\varphi(n) = \begin{cases} A_0 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}$, where

$$A_0 = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad \text{with respect to the basis} \quad \{e_{00}, \alpha e_{11} - e_{21}\}$$

$\hat\varphi \in \mathcal{GM}(S_{\boldsymbol\lambda})$ if and only if $b = c = 0$ and $a = d$.

Examples of General multipliers



Example

$\hat{\psi}\colon \mathbb{N}_0 \to \mathbf{B}(\mathcal{N}(S_{\boldsymbol{\lambda}}^*))$ be defined as $\hat{\psi}(k) = \begin{cases} A_k & \text{if } k < 2 \\ 0 & \text{if } k \geqslant 2 \end{cases}$, where

$$A_k = \begin{pmatrix} a_k & c_k \\ b_k & d_k \end{pmatrix}, \quad \text{with respect to the basis} \quad \{e_{00}, \alpha e_{11} - e_{21}\}$$

Then $\hat{\psi}$ is a generalized multiplier if and only if

$$A_0 = \begin{pmatrix} a_0 & 0 \\ \frac{d_1 - a_1}{\alpha} & d_0 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} a_1 & (a_0 - d_0)\alpha \\ 0 & d_1 \end{pmatrix}.$$

Commutant

Definition

For a given operator $A \in \mathbf{B}(\mathcal{H})$ let $\hat{\varphi}_A \colon \mathbb{N}_0 \to \mathbf{B}(\mathcal{N}(T^*))$ be the following sequence

$$\hat{\varphi}_A(m) = P_{\mathcal{N}(T^*)} L^m A|_{\mathcal{N}(T^*)}, \quad m \in \mathbb{N}_0.$$

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Theorem

Let $T \in \mathbf{B}(\mathcal{H})$ be left-invertible and analytic. Assume that $A \in \mathbf{B}(\mathcal{H})$ commutes with T. Then $\hat{\varphi}_A \in \mathcal{GM}(T)$ and $A = U^* M_{\hat{\varphi}_A} U$.

Balanced weighted shifts

A directed tree $\mathscr{T} = (V, E)$

is leafless iff $\mathrm{Chi}(v) \neq \emptyset$ for all $v \in V$;

is rooted iff it has a root;

then define $|v|$ is a generation of $v \in V$ iff $v \in \mathrm{Chi}^{\langle |v| \rangle}(\mathrm{root})$;

Definition

Let $\mathscr{T} = (V, E)$ be a countably infinite rooted and leafless directed tree, $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ} \subseteq (0, \infty)$ and let $S_{\boldsymbol{\lambda}}$ be a bounded weighted shift on $\mathscr{T}$ with weights $\boldsymbol{\lambda}$. If

$$\|S_{\boldsymbol{\lambda}} e_u\| = \|S_{\boldsymbol{\lambda}} e_v\| \text{ for every } u, v \in V \text{ such that } |u| = |v|,$$

then we say that $S_{\boldsymbol{\lambda}}$ is balanced.

Separated basis

Let us define the k-th generation of vertices as the set

$$V_k := \{v \in V : |v| = k\}, \text{ for some } k \in \mathbb{N}_0.$$

Functions acting on k-th generation of vertices forms the set

$$\ell^2(V_k) = \{f \in \ell^2(V) : f(u) = 0 \text{ if } |u| \neq k\}, \quad k \in \mathbb{N}_0.$$

Lemma

Let $\mathscr{T} = (V, E)$ be a countably infinite rooted and leafless directed tree, and $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ} \subseteq (0, \infty)$. Let $S_{\boldsymbol{\lambda}} \in \mathbf{B}(\ell^2(V))$ be left-invertible. Then there exists an orthonormal basis $\{e_j'\}_{j \in J}$ of $\mathcal{N}(S_{\boldsymbol{\lambda}}^*)$ such that for every $j \in J$ vector e_j' belongs to the space $\ell^2(V_{k_j})$ for some $k_j \in \mathbb{N}_0$.

H^∞ space

Definition

For given $\beta \in (0, +\infty)^{\mathbb{N}_0}$ we set

$$H^\infty(\beta) := \left\{a \in \mathbb{C}^{\mathbb{N}_0} : a * b \in \ell^2(\beta) \text{ for every } b \in \ell^2(\beta)\right\},$$

where for any sequence $\beta = \{\beta_n\}_{n=0}^\infty \subset (0, +\infty)$ the space $\ell^2(\beta)$ is the weighted ℓ^2 space

$$\left\{\{a_n\}_{n=0}^\infty \in \mathbb{C}^{\mathbb{N}_0} : \sum_{n=0}^\infty |a_n|^2 \beta_n < \infty\right\}.$$

Commutant of balanced weighted shift

Theorem

Let $\mathscr{T} = (V, E)$ be a countably infinite rooted and leafless directed tree, and $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ} \subseteq (0, \infty)$. Assume that $S_{\boldsymbol{\lambda}} \in \mathbf{B}(\ell^2(V))$ is balanced and left-invertible. Assume also that $\dim \mathcal{N}(S_{\boldsymbol{\lambda}}^*) < \infty$ and $\{e_j'\}_{j \in J}$ is a separated basis of $\mathcal{N}(S_{\boldsymbol{\lambda}}^*)$. Then

$$\mathcal{GM}(S_{\boldsymbol{\lambda}}) = \Big\{\hat{\varphi} \colon \mathbb{N}_0 \to \mathbf{B}\big(\mathcal{N}(S_{\boldsymbol{\lambda}}^*)\big) \Big| \; \big\{\langle \hat{\varphi}(n) e_j', e_i' \rangle\big\}_{n=0}^{\infty} \in H^\infty\big(\{\|S^n e_{\mathrm{root}}\|^2\}_{n=0}^{\infty}\big), \; i, j \in J\Big\}.$$

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Thank you for your attention!