Perturbation bounds for Mostow's decomposition

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Mostow's decomposition

Polar decomposition

Any invertible matrix A can be written uniquely as $A = UP$, U unitary, P positive definite

Mostow's decomposition

Every invertible matrix Z can be uniquely factorized as

$$Z = W\, e^{iK} e^{S},$$

where W is a unitary matrix, S is a real symmetric matrix and K is a real skew symmetric matrix.

$$e^{iK} = P_1, e^{S} = P_2$$

P_1, P_2 are positive definite
Moreover, P_1 is circular (i.e. $\overline{P_1} P_1 = I$)
So

$$Z = W\ P_1\ P_2$$

Perturbation bounds

$\mathscr{X}, \mathscr{Y}$: normed spaces (or their open subsets) or Lie groups

$$f : \mathscr{X} \to \mathscr{Y}$$

$$\|f(u) - f(v)\| \leq C\|u - v\|$$

$$Z = W\ P_1\ P_2$$

$$Z \mapsto W, \quad Z \mapsto P_1, \quad Z \mapsto P_2$$

Derivative

If f is differentiable at $u \in \mathscr{X}$, then for every $v \in \mathscr{X}$ ($v \in$ Lie Algebra)

$$Df(u)(v) = \left.\frac{d}{dt}\right|_{t=0} f(u+tv).$$

$$\left(Df(u)(uv) = \left.\frac{d}{dt}\right|_{t=0} f(ue^{tv}). \right)$$

Mean value theorem

Let $f : \mathscr{X} \to \mathscr{Y}$ be a differentiable map. Let $u, v \in \mathscr{X}$ and let L be the line segment joining them. Then

$$\|f(u) - f(v)\| \leq \|u - v\| \sup_{w \in L} \|Df(w)\|.$$

Taylor's theorem

Let $f : \mathscr{X} \to \mathscr{Y}$ be a $(p+1)$-times differentiable map. For $u \in \mathscr{X}$ and for small h,

$$\left\| f(u+h) - f(u) - \sum_{k=1}^{p} \frac{1}{k!} D^k f(u)(h, \ldots, h) \right\| = O(\|h\|^{p+1}).$$

From here, first order perturbation bounds can be found.

$$\|f(u+h) - f(u)\| \leq \|Df(u)\| \, \|h\| + O(\|h\|^2).$$

Notations

$\mathbb{M}(n, \mathbb{C})$: space of $n \times n$ complex matrices

$\mathbb{GL}(n, \mathbb{C})$: set of invertible matrices

$\mathbb{P}(n, \mathbb{C})$: set of $n \times n$ positive definite matrices.

$\mathbb{H}(n, \mathbb{C})$: space of $n \times n$ Hermitian matrices

$\mathbb{SH}(n, \mathbb{C})$: space of $n \times n$ skew-Hermitian matrices

$\mathbb{U}(n, \mathbb{C})$: set of $n \times n$ unitary matrices.

Notations

- $\|A\|$: operator norm of A
  If $s_1 \geq s_2 \geq \cdots \geq s_n \geq 0$ are singular values of A, then $\|A\| = s_1$.
- $|||A|||$: unitarily invariant norm of A

$$|||A||| = |||UAV||| \text{ for all } U, V \text{ unitary}$$

- $\overline{A}$: complex conjugate of A
- $$|||A||| = |||A^*||| = |||\, \overline{A} \,|||$$
- $$|||ABC||| \leq \|A\| \; |||B||| \; \|C\|$$
- Let $\mathscr{W}$ be a subspace of $(\mathbb{M}(n, \mathbb{C}), ||| \cdot |||)$ and let $\mathcal{T} : \mathscr{W} \to \mathbb{M}(n, \mathbb{C})$ be a linear map. Then

$$|||\mathcal{T}||| \quad = \quad \sup\{|||\mathcal{T}(X)||| : |||X||| = 1\}.$$

Matrix Factorization

- $\mathbb{A}, \mathbb{A}_1, \mathbb{A}_2$: classes of matrices (open subsets of $\mathbb{M}(n, \mathbb{C})$ or Lie groups)
- Every $A \in \mathbb{A}$ has a unique factorization

$$A = A_1 A_2$$

- The decomposition gives a map $\varrho : \mathbb{A} \to \mathbb{A}_1 \times \mathbb{A}_2$

$$\varrho(A) = (\varrho_1(A), \varrho_2(A)) = (A_1, A_2)$$

- To study variation of A_1 , A_2 with A, it is natural to study the derivatives $D\varrho_1(A), D\varrho_2(A)$.
- The maps ϱ_1, ϱ_2 are complicated to describe.

Bhatia and Mukherjea (1994)

- Instead, consider the inverse map

  $$\tau : \mathbb{A}_1 \times \mathbb{A}_2 \to \mathbb{A}$$

  defined as

  $$\tau(A_1, A_2) = A_1 A_2 = A$$

- τ is a product map so computing the derivative is easy.

- Inverse function theorem: Let $\mathbb{A}$ be an open subset of $\mathbb{R}^n$ and let $f : \mathbb{A} \to \mathbb{R}^n$ be a continuously differentiable map. Let $p \in \mathbb{A}$ such that $\det Df(p) \neq 0$. Then there is an open set U containing p and an open set V containing $f(p)$ such that $f : U \to V$ has a differentiable inverse $f^{-1} : V \to U$ and for $y \in V$,

  $$D(f^{-1})(y) = \left[Df(f^{-1}(y)) \right]^{-1}.$$

Inverse Function Theorem

Let $f : N \to M$ be a continuously differentiable map between two manifolds of the same dimension, and $p \in N$. Suppose for some charts (U, ϕ) around $p \in N$ and (V, ψ) around $f(p)$ in M, $f(U) \subset V$.Then f is locally invertible at p if the Jacobian determinant is nonzero.

- $$\tau : \mathbb{A}_1 \times \mathbb{A}_2 \to \mathbb{A}$$

  $$D\tau(A_1, A_2) : T_{A_1}\mathbb{A}_1 + T_{A_2}\mathbb{A}_2 \to T_A\mathbb{A},$$

  where $T_A\mathbb{A}$ is the tangent space to $\mathbb{A}$ at the point A
- When $\mathbb{A}$ is an open set, then $T_A\mathbb{A} = \mathbb{M}(n, \mathbb{C})$ for every $A \in \mathbb{A}$. All tangential vectors at A are of the form $A + tB$, $B \in \mathbb{M}(n, \mathbb{C})$.
- When $\mathbb{A}$ is a Lie group, then $T_I\mathbb{A}$ is the Lie algebra corresponding to this Lie group. The tangent space at any other point is $A \cdot T_I\mathbb{A}$. The tangent vectors at A are written as Ae^{tB}, where B is from the Lie algebra.

Polar decomposition

$A \in \mathbb{GL}(n, \mathbb{C})$ can be written uniquely as $A = UP$, U unitary, P positive definite

$$\varrho_1 : \mathbb{GL}(n, \mathbb{C}) \to \mathbb{U}(n, \mathbb{C})$$

$$\varrho_2 : \mathbb{GL}(n, \mathbb{C}) \to \mathbb{P}(n, \mathbb{C})$$

$$T_A \mathbb{GL}(n, \mathbb{C}) = \mathbb{M}(n, \mathbb{C}),\ T_U \mathbb{U}(n, \mathbb{C}) = U\, \mathbb{SH}(n, \mathbb{C}),\ T_P \mathbb{P}(n, \mathbb{C}) = \mathbb{H}(n, \mathbb{C})$$

For $S \in \mathbb{SH}(n, \mathbb{C}), H \in \mathbb{H}(n, \mathbb{C})$,

$$\begin{aligned} D\tau(U, P)(US, H) &= \left.\frac{d}{dt}\right|_{t=0} \tau(Ue^{tS}, P + tH) \\ &= \left.\frac{d}{dt}\right|_{t=0} Ue^{tS}(P + tH) \\ &= USP + UH. \end{aligned}$$

Polar decompostion

For all $X \in \mathbb{M}(n, \mathbb{C})$, we want to find $D\varrho(A)(X)$.
Suppose

$$D\varrho(A)(UX) = (US, H) \text{ for some } S \in \mathbb{SH}(n, \mathbb{C}), H \in \mathbb{H}(n, \mathbb{C})$$

$$UX = D\tau(U, P)(US, H) = USP + UH$$

$$X = SP + H$$

In this case, S and H can be found explicitly. But even if they can't be found, this gives adequate information to get bounds on $D\varrho_1(A)$ and $D\varrho_2(A)$.

Observation

Tangent spaces at U and P give

$$\mathbb{M}(n,\mathbb{C}) = \mathbb{H}(n,\mathbb{C}) + \mathbb{SH}(n,\mathbb{C}).$$

If $\mathscr{P}_1$ and $\mathscr{P}_2$ are the corresponding projection operators , then

$$\mathscr{P}_1(A) = \frac{A - A^*}{2}$$

$$\mathscr{P}_2(A) = \frac{A + A^*}{2}$$

So

$$|||\mathscr{P}_1||| = |||\mathscr{P}_2||| = 1.$$

Finding S and H

$$X = SP + H, X \in \mathbb{M}(n, \mathbb{C}), S \in \mathbb{SH}(n, \mathbb{C}), H \in \mathbb{H}(n, \mathbb{C})$$

$$X^* = -PS + H$$

Subtract

$$X - X^* = SP + PS$$

This is a special case of well known Sylvester's equation.

Sylvester equation

$$AX - XB = Y$$

If $\sigma(A)$ and $\sigma(B)$ are spectra of A and B such that $\sigma(A) \cap \sigma(B) = \phi$, then Sylvester's equation has a unique solution X for every Y.

If $\sigma(A)$ is contained in the open right half-plane and $\sigma(B)$ is contained in the open left half-plane, then

$$X = \int_0^\infty e^{-tA} Y e^{tB} dt.$$

(Note: If $a - b \neq 0$, then $ax - xb = y$ has a unique solution $x = \frac{y}{a-b}$. If Re $(b-a) < 0$, then $\int_0^\infty e^{t(b-a)} dt$ is convergent and has the value $\frac{1}{a-b}$. In this case, the solution of $ax - xb = y$ can be expressed as $x = \int_0^\infty e^{t(b-a)} y dt$.)

Bound on $D\varrho_1(A)$

$$2i \operatorname{Im} X = X - X^* = SP + PS$$

$$S = 2i \int_0^\infty e^{-tP} \operatorname{Im} X \, e^{-tP} dt$$

Thus

$$|||S||| \leq 2 \left(\int_0^\infty \|e^{-tP}\|^2 dt \right) ||| \operatorname{Im} X|||$$

Easy to compute

$$\int_0^\infty \|e^{-tP}\|^2 dt = \frac{\|P^{-1}\|}{2} = \frac{\|A^{-1}\|}{2}$$

$$|||S||| \leq \|A^{-1}\| \, ||| \operatorname{Im} X||| \leq \|A^{-1}\| \, |||X|||$$

Hence

$$|||D\varrho_1(A)(UX)||| \leq \|A^{-1}\| \, |||X|||$$

For $X = i\, I/|||I|||$, this is an equality.

Bound on $D_{\varrho_2}(A)$

$$X = SP + H$$

$$\begin{aligned} |||H||| &\leq |||X||| + |||SP||| \\ &\leq |||X||| + |||S|||\,\|P\| \\ &\leq |||X||| + \|A^{-1}\|\,|||X|||\,\|P\| \\ &= [1 + \mathrm{cond}(A)]|||X|||, \end{aligned}$$

where $\mathrm{cond}(A) = \|A\|\,\|A^{-1}\|$.
So

$$|||D_{\varrho_2}(A)||| \leq 1 + \mathrm{cond}(A).$$

Perturbation bounds

First order perturbation bounds can be found using Taylor's theorem.
Let $\tilde{A}$ represent a perturbation of A ($\tilde{A}$ is in a neighbourhood of A). Let $\tilde{A} = \tilde{U}\tilde{P}$. Then Taylor's theorem gives

$$|||\tilde{U} - U||| \leq \|A^{-1}\| \; |||\tilde{A} - A||| + O(|||\tilde{A} - A|||^2).$$

This is usually represented as

$$|||\tilde{U} - U||| \lesssim \|A^{-1}\| \; |||\tilde{A} - A|||.$$

Similarly,

$$|||\tilde{P} - P||| \lesssim [1 + \mathrm{cond}(A)] \; |||\tilde{A} - A|||.$$

Mostow's decomposition

$$Z = W \; P_1 \; P_2$$

Recently, Bhatia (2013) gave another proof of Mostow's decomposition theorem, giving explicitly what these factors are.

Related to geometric mean of two positive definite matrices.

Geometric mean

For $A, B \in \mathbb{P}(n, \mathbb{C})$ their geometric mean is defined as

$$A \# B = A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^{1/2} A^{1/2}$$

- It is the unique positive solution of the *Riccati equation*

$$X A^{-1} X = B$$

- $$A \# B = \max \left\{ X : X = X^*, \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq O \right\}$$

- $$A \# B = A(A^{-1}B)^{1/2} = (AB^{-1})^{1/2} B,$$

  where $(A^{-1}B)^{1/2}$ and $(AB^{-1})^{1/2}$ are the unique square roots of $A^{-1}B$ and AB^{-1}, respectively, with positive eigenvalues.

Mostow's decomposition

$$A = Z^* Z$$

$A \# \overline{A}$ is positive definite

$$(A \# \overline{A})^{1/2} = P_2 = e^S, \; S \text{ real symmetric}$$

$$P_1 = (e^{-S} A e^{-S})^{1/2}$$

P_1 is positive definite and circular

$$P_1 = e^{iK}, \; K \text{ real skew symmetric}$$

$$W = Z e^{-S} e^{-iK}$$

Then W is unitary

Derivative of geometric mean (–, Mishra (2017))

Let $G : \mathbb{P}(n, \mathbb{C}) \times \mathbb{P}(n, \mathbb{C}) \to \mathbb{P}(n, \mathbb{C})$ be the map defined as

$$G(A, B) = A \# B = A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^{1/2} A^{1/2}.$$

Since $A \mapsto A^{1/2}$ is a differentiable function on $\mathbb{P}(n, \mathbb{C})$, G is a differentiable map.

$$DG(A, B)(X, Y) = \left. \frac{d}{dt} \right|_{t=0} G(A + tX, B + tY) \text{ for all } X, Y \in \mathbb{H}(n, \mathbb{C}).$$

Derivative of geometric mean

$$X, Y \in \mathbb{H}(n, \mathbb{C})$$

For sufficiently small t

$$G(A+tX, B+tY)(A+tX)^{-1}G(A+tX, B+tY) = B+tY.$$

Differentiating with respect to t at 0, we get

$$\begin{aligned}(DG(A,B)(X,Y))\,A^{-1}G(A,B) - G(A,B)(A^{-1}XA^{-1})G(A,B)\\ +G(A,B)A^{-1}\,(DG(A,B)(X,Y)) = Y.\end{aligned}$$

(using $DA^{-1}(X) = -A^{-1}XA^{-1}$)

Put $D = DG(A,B)(X,Y)$ and $C = A^{-1}G(A,B) = (A^{-1}B)^{1/2}$.

$$C^*D + DC = Y + C^*XC.$$

Derivative of geometric mean

For $X, Y \in \mathbb{H}(n, \mathbb{C})$

$$DG(A,B)(X,Y) = \int_0^\infty e^{-tC^*}(Y + C^*XC)e^{-tC}dt$$
$$= \int_0^\infty e^{-t(BA^{-1})^{1/2}}(Y + (BA^{-1})^{1/2}X(A^{-1}B)^{1/2})e^{-t(A^{-1}B)^{1/2}}dt$$

$$\begin{aligned} |||DG(A,B)||| &= \sup\{|||DG(A,B)(X,Y)||| : |||(X,Y)||| = 1\}, \\ &\quad \text{where } |||(X,Y)||| = \max\{|||X|||, |||Y|||\} \\ &\le \left(\int_0^\infty \|e^{-t(A^{-1}B)^{1/2}}\|^2 dt\right)\left(1 + \|(A^{-1}B)^{1/2}\|^2\right). \end{aligned}$$

Mostow's decomposition

$$Z = WP_1 P_2$$

$\varrho_2 : Z \mapsto P_2$:

$$P_2 = (Z^* Z \# \overline{Z^* Z})^{1/2}$$

Then

$$\varrho_2 = f \circ G \circ g \circ h,$$

where $f : t \mapsto t^{1/2}$, $g : A \mapsto (A, \overline{A})$, $h : Z \mapsto Z^* Z$.

Chain rule:

$$D\varrho_2(Z) = Df(Z^* Z \# \overline{Z^* Z}) \circ DG(Z^* Z, \overline{Z^* Z}) \circ Dg(Z^* Z) \circ Dh(Z)$$

- $$|||Df(A)||| \leq \frac{1}{2} \|A^{-1}\|^{1/2}$$
- $\|A \# B\| \leq \|A\|^{1/2} \|B\|^{1/2}$

Bound for $|||D\varrho_2(Z)|||$

$$|||D\varrho_2(Z)(A)||| \leq \frac{1}{2}\|Z^{-1}\|\ |||DG(Z^*Z, \overline{Z^*Z})(Z^*A+AZ, \overline{Z^*A+AZ})|||$$

Let $C = (Z^*Z)^{-1}\left(Z^*Z\#\overline{Z^*Z}\right) = \left((Z^*Z)^{-1}\overline{Z^*Z}\right)^{1/2}$. Then $\|C\| \leq \mathrm{cond}(Z)^2$.

$$|||D\varrho_2(Z)(A)||| \leq \frac{1}{2}\|Z^{-1}\|\beta(Z)\left(1+\|C\|^2\right)|||Z^*A+AZ|||,$$

where $\beta(Z) = \int_0^\infty \|e^{-t\left((Z^*Z)^{-1}\overline{Z^*Z}\right)^{1/2}}\|^2 dt$

$$|||D\varrho_2(Z)(A)||| \leq \beta(Z)\ \mathrm{cond}(Z)\left(1+\ \mathrm{cond}(Z)^4\right)\ |||A|||$$

Mostow's decomposition

$\varrho_1 : Z \mapsto P_1$, $\varrho_0 : Z \mapsto W$:
$\mathbb{P}_{circ}$: the set of circular positive definite matrices.

$$P_1 \in \mathbb{P}_{circ}$$

$$\varrho(Z) = (\varrho_0(Z), \varrho_1(Z), \varrho_2(Z))$$

We are interested in $D\varrho_1(Z)$ and $D\varrho_0(Z)$.
Tangent spaces?
At W, it is $W\,\mathbb{SH}(n, \mathbb{C})$.
At P_1?

Tangent space at I

Consider any smooth curve $\gamma(t)$ in $\mathbb{P}_{circ}$, with $\gamma(0) = I$. We have

$$\overline{\gamma(t)}\gamma(t) = I$$

and

$$\gamma(t)^* = \gamma(t).$$

Differentiating at $t = 0$ gives that $\overline{\gamma'(0)} + \gamma'(0) = 0$ so that

$$\gamma'(0) \text{ is a purely imaginary matrix.}$$

Also, $\gamma'(0)$ is Hermitian.
So

$$T_I\mathbb{P}_{circ} = i\mathbb{SH}(n, \mathbb{R}),$$

where $\mathbb{SH}(n, \mathbb{R})$: space of $n \times n$ real skew symmetric matrices.

Tangent space at P_1

To calculate $T_{P_1}\mathbb{P}_{circ}$, consider $\gamma(t)$ in $\mathbb{P}_{circ}$, with $\gamma(0) = P_1$. So

$$\overline{\gamma'(0)}P_1 + \overline{P_1}\gamma'(0) = 0.$$

For every $J \in \mathbb{SH}(n, \mathbb{R})$, the matrix $P_1^{1/2} iJP_1^{1/2}$ satisfies the above equation. A count on dimensions shows that the tangent space at any point P_1 is given by $iP_1^{1/2}\mathbb{SH}(n, \mathbb{R})P_1^{1/2}$.

$$D\varrho(Z) : \mathbb{M}(n,\mathbb{C}) \to W\,\mathbb{SH}(n,\mathbb{C}) \oplus i\,P_1^{1/2}\mathbb{SH}(n,\mathbb{R})P_1^{1/2} \oplus \mathbb{H}(n,\mathbb{R})$$

Suppose

$$D\varrho(Z)(A) = (WX_0, i\,P_1^{1/2}Y_1P_1^{1/2}, Y_2),$$

where $X_0 \in \mathbb{SH}(n,\mathbb{C})$, $Y_1 \in \mathbb{SH}(n,\mathbb{R})$ and $Y_2 \in \mathbb{H}(n,\mathbb{R})$.

$$D\varrho(Z)(A) = (WX_0, iP_1^{1/2} Y_1 P_1^{1/2}, Y_2),$$

$$D\tau(W, P_1, P_2)(WX_0, iP^{1/2} Y_1 P^{1/2}, Y_2) = A.$$

By definition,

$$D\tau(W, P_1, P_2)(WX_0, iP^{1/2} Y_1 P^{1/2}, Y_2)$$

$$= \frac{d}{dt}\bigg|_{t=0} \tau(We^{tX_0}, P_1^{1/2} e^{itY_1} P_1^{1/2}, P_2 + tY_2)$$

$$= \frac{d}{dt}\bigg|_{t=0} We^{tX_0} P_1^{1/2} e^{itY_1} P_1^{1/2} (P_2 + tY_2)$$

$$= WX_0 P_1 P_2 + WP_1^{1/2}(iY_1)P_1^{1/2} P_2 + WP_1 Y_2.$$

$$WX_0P_1P_2 + WP_1^{1/2}(iY_1)P_1^{1/2}P_2 + WP_1Y_2 = A,$$

where

$$X_0^* = -X_0,\ \overline{Y_1} = Y_1,\ Y_1^t = -Y_1,\ \overline{Y_2} = Y_2,\ Y_2^t = Y_2$$

i.e.

$$X_0 + P_1^{1/2}(iY_1)P_1^{-1/2} = (W^*A - P_1Y_2)(P_1P_2)^{-1}$$

Taking adjoint,

$$-X_0 + P_1^{-1/2}(iY_1)P_1^{1/2} = (P_1P_2)^{-1}(A^*W - Y_2P_1)$$

Add.

$$(P_1^{1/2}(iY_1)P_1^{1/2})P_1^{-1} + P_1^{-1}(P_1^{1/2}(iY_1)P_1^{1/2})$$
$$= \mathrm{Re}\left((W^*A - P_1Y_2)(P_1P_2)^{-1}\right).$$

$$P_1^{1/2}(iY_1)P_1^{1/2} \;=\; \int_0^\infty e^{-tP_1^{-1}} \mathrm{Re}\, \left((W^*A - P_1 Y_2)(P_1 P_2)^{-1} \right) e^{-tP_1^{-1}} dt.$$

We get

$$\begin{aligned}
|||D\varrho_1(Z)(A)||| &= |||P_1^{1/2}(iY_1)P_1^{1/2}||| \\
&\le \left(\int_0^\infty \|e^{-tP_1^{-1}}\|^2 dt \right) |||\mathrm{Re}\, \left((W^*A - P_1 Y_2)(P_1 P_2)^{-1} \right. \\
&\le \frac{\|P_1\|}{2} |||(W^*A - P_1 Y_2)(P_1 P_2)^{-1}||| \\
&\le \frac{\mathrm{cond}(P_1)\, \|P_2^{-1}\|}{2} (1 + \|P_1\| |||Y_2|||)\ |||A|||.
\end{aligned}$$

$|||D\varrho_0(Z)(A)|||$

Similarly, we obtain

$$X_0 P_1 + P_1 X_0 = 2i \operatorname{Im}\left((W^* A - P_1 Y_2) P_2^{-1}\right),$$

which gives

$$\begin{aligned} & D\varrho_0(Z)(A) = X_0 \\ = & \int_0^\infty e^{-tP_1} \operatorname{Im}\left((W^* A - P_1 Y_2) P_2^{-1}\right) e^{-tP_1} dt. \end{aligned}$$

So

$$\begin{aligned} |||D\varrho_0(Z)(A)||| &= |||X_0||| \\ &\leq \left(\int_0^\infty \|e^{-tP_1}\|^2 dt\right) |||(W^* A - P_1 Y_2) P_2^{-1}||| \\ &\leq \frac{\|P_1^{-1}\| \, \|P_2^{-1}\|}{2} (1 + \|P_1\| |||Y_2|||) \, |||A|||. \end{aligned}$$

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THANK YOU!