Weighted composition operators in L^2-spaces

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- $(X, \mathscr{A}, \mu)$ is a σ-finite measure space
- $\phi\colon X \to X$ is $\mathscr{A}$-measurable, $w\colon X \to \mathbb{C}$ is $\mathscr{A}$-measurable

Define $C_{\phi,w}\colon L^2(\mu) \supseteq \mathcal{D}(C_{\phi,w}) \to L^2(\mu)$ by

$$\mathcal{D}(C_\phi) = \{f \in L^2(\mu)\colon \int |f \circ \phi|^2 d\mu_w < \infty\},$$
$$C_\phi f = w\,(f \circ \phi), \quad f \in \mathcal{D}(C_{\phi,w}).$$

Proposition

$C_{\phi,w}$ is well-defined if and only if $\mu_w \circ \phi^{-1}$ is absolutely continuosu with respect to μ, where

- $\mu_w \circ \phi^{-1}(\varDelta) := \mu_w\big(\phi^{-1}(\varDelta)\big)$, $\varDelta \in \mathscr{A}$,
- $\mu_w(\varDelta) = \int_\varDelta |w|^2 d\mu$.

Subclasses

- multiplication operators in L^2-spaces,
- composition and partial composition operators in L^2-spaces,
- unilateral/bilateral weighted shifts,
- adjoints of unilateral/bilateral weighted shifts,
- weighted shifts on directed trees.

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Example

consider the following

- $X = \mathbb{Z}_+$, $\mathscr{A} = 2^X$
- $\mu(n) = 1$ for $n \in \mathbb{N}$ and $\mu(0) = 0$
- $\phi\colon X \to X$ given by

$$\phi(n) = \begin{cases} n-1 & \text{if } n \in \mathbb{N} \\ 0 & \text{if } n = 0 \end{cases}$$

- $w\colon X \to \mathbb{C}$ given by

$$w(n) = \begin{cases} 0 & \text{if } n \in \{0,1\} \\ 1 & \text{if } n \geqslant 2 \end{cases}$$

then

- $C_{\phi,w}$ is **isometric**
- C_ϕ is **not well-defined**

Remark

The following restrictive assumptions have been made in a literature:

- $(X, \mathscr{A}, \mu)$ is complete
- C_ϕ is well-defined
- C_ϕ is densely defined
- $w \geqslant 0$ a.e. $[\mu]$

Remark

- in general, $C_{\phi,w} \neq M_w C_\phi$ even if C_ϕ is well-defined.

Radon-Nikodym derivative

- $\mu_w \circ \phi^{-1} \ll \mu$

$$h_{\phi,w} = \frac{d\mu_w \circ \phi^{-1}}{d\mu}$$

For every $f\colon X \to \overline{\mathbb{R}}_+$ or $f\colon X \to \mathbb{C}$ such that $f \circ \phi \in L^2(\mu_w)$ we have

$$\int_X f \circ \phi \, d\mu_w = \int_X f \, h_{\phi,w} d\mu$$

- $h_\phi := h_{\phi,\mathbf{1}}$, where $\mathbf{1} := \chi_X$

Proposition

Assume that $C_{\phi,w}$ is well-defined. Then:

(i) $\mathcal{D}(C_{\phi,w}) = L^2\big((1+h_{\phi,w})d\mu\big)$.

(ii) $C_{\phi,w}$ is densely defined in $L^2(\mu)$ if and only if $h_{\phi,w} < \infty$ a.e. $[\mu]$ if and only if $\mu_w \circ \phi^{-1}$ is σ-finite.

(iii) $\overline{\mathcal{D}(C_{\phi,w})} = \chi_{\{h_{\phi,w}<\infty\}} L^2(\mu)$.

(iv) $C_{\phi,w}$ is closed.

(v) $C_{\phi,w} \in \mathbf{B}\big(L^2(\mu)\big)$ if and only if $h_{\phi,w} \in L^\infty(\mu)$; if $C_{\phi,w} \in \mathbf{B}\big(L^2(\mu)\big)$, then $\|C_{\phi,w}\|^2 = \|h_{\phi,w}\|_{L^\infty(\mu)}$.

(v) $\mathcal{N}(C_{\phi,w}) = \chi_{\{h_{\phi,w}=0\}} L^2(\mu)$.

Proposition

(i) if C_ϕ is well-defined, then $C_{\phi,w}$ is well-defined and $M_w C_\phi \subseteq C_{\phi,w}$.

(ii) if $w \neq 0$ a.e. $[\mu]$ and $C_{\phi,w}$ is well-defined, then C_ϕ is well-defined.

(iii) if $C_{\phi,w}$ is well-defined and has dense range, then C_ϕ is well-defined.

Theorem

Assume that C_ϕ is well-defined. Then $C_{\phi,w}$ is well-defined and tfae:

(i) $M_w C_\phi = C_{\phi,w}$.

(ii) there exists $c \in \mathbb{R}_+$ such that $h_\phi \leqslant c(1 + h_{\phi,w})$ a.e. $[\mu]$.

Proposition

(i) if $C_\phi \in \mathbf{B}\big(L^2(\mu)\big)$, then $C_{\phi,w}$ is well-defined and $M_w C_\phi = C_{\phi,w}$.

(ii) if μ-ess inf $|w| > 0$ and $C_{\phi,w}$ is well-defined, then C_ϕ is well-defined and $M_w C_\phi = C_{\phi,w}$.

(iii) if $M_w \in \mathbf{B}\big(L^2(\mu)\big)$ and C_ϕ is well-defined, then $C_{\phi,w}$ is well-defined and $h_{\phi,w} \leqslant \|M_w\|^2 h_\phi$.

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Theorem

Assume that C_ϕ is well-defined. Then $C_{\phi,w}$ is well-defined and tfae:

(i) $M_w C_\phi$ is a closed operator.

(ii) there exists $c \in \mathbb{R}_+$ such that $h_\phi \leqslant c(1 + h_{\phi,w})$ a.e. $[\mu]$ on $\{h_\phi < \infty\}$.

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Conditional expectation

- $h_{\phi,w} < \infty$ a.e. $[\mu]$

If $f\colon X \to \mathbb{R}_+$ is $\mathscr{A}$-measurable, then $\mathsf{E}_{\phi,w}(f)$ is the unique $\phi^{-1}(\mathscr{A})$-measurable function on X such that

$$\int_{\phi^{-1}(\varDelta)} f d\mu = \int_{\phi^{-1}(\varDelta)} \mathsf{E}_{\phi,w}(f) d\mu, \quad \varDelta \in \mathscr{A}.$$

We call $\mathsf{E}_{\phi,w}(f)$ the conditional expectation of f with respect to $\phi^{-1}(\mathscr{A})$.

- $\mathsf{E}_{\phi,w}(f)$ makes sense whenever $f\colon X \to \overline{\mathbb{R}}_+$ or $f \in L^p(\mu)$, $p \in [1, \infty]$
- the mapping $L^2(\mu_w) \ni f \mapsto \mathsf{E}_{\phi,w}(f) \in L^2(\mu_w)$ defines an orthogonal projection

Proposition

Assume that C_ϕ is densely defined. Then $C_{\phi,w}$ is well-defined and

$$h_{\phi,w} = h_\phi \, \mathsf{E}_\phi\big(|w|^2\big) \circ \phi^{-1} \quad \text{a.e. } [\mu].$$

Proposition

Assume that $w \neq 0$ a.e. $[\mu]$ and $C_{\phi,w}$ is densely defined. Then C_ϕ is well-defined and

$$h_\phi = h_{\phi,w} \, \mathsf{E}_{\phi,w}\Big(\frac{1}{|w|^2}\Big) \circ \phi^{-1} \quad \text{a.e. } [\mu].$$

Proposition

Assume that $w \neq 0$ a.e. $[\mu]$ and $C_{\phi,w}$ and C_ϕ is densely defined. Then the following are hold:

(i) $\{h_{\phi,w} > 0\} = \{h_\phi > 0\}$ a.e. $[\mu]$,

(ii) $\mathsf{E}_\phi\big(|w|^2\big) \circ \phi^{-1} \, \mathsf{E}_{\phi,w}\big(\frac{1}{|w|^2}\big) \circ \phi^{-1} = \chi_{\{h>0\}}$ a.e. $[\mu]$,

(iii) $\mathsf{E}_\phi\big(|w|^2\big) \, \mathsf{E}_{\phi,w}\big(\frac{1}{|w|^2}\big) = 1$ a.e. $[\mu]$

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Assume that $w \neq 0$ a.e. $[\mu]$ and $C_{\phi,w}$ and C_ϕ is densely defined. Then the following are hold:

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(iii) $\mathsf{E}_\phi\big(|w|^2\big) \,\mathsf{E}_{\phi,w}\big(\frac{1}{|w|^2}\big) = 1$ a.e. $[\mu]$

Theorem

Assume that $C_{\phi,w}$ is densely defined. Then the following equalities hold:

$$\mathcal{D}(C_{\phi,w}^*) = \{f \in L^2(\mu) \colon h_{\phi,w} \cdot \mathsf{E}_{\phi,w}(f_w) \circ \phi^{-1} \in L^2(\mu)\},$$
$$C_{\phi,w}^* f = h_{\phi,w} \cdot \mathsf{E}_{\phi,w}(f_w) \circ \phi^{-1}, \quad f \in \mathcal{D}(C_{\phi,w}^*),$$

where

$$f_w = \chi_{\{w \neq 0\}} \frac{f}{w}.$$

Moreover, we have

(i) $\mathcal{N}(C_{\phi,w}^*) = \{f \in L^2(\mu) \colon \mathsf{E}_{\phi,w}(f_w) = 0 \text{ a.e. } [\mu_w]\}$,

(ii) $\chi_{w=0} L^2(\mu) \subseteq \mathcal{N}(C_{\phi,w}^*)$,

(iii) $\mathcal{D}(C_{\phi,w}^*) = \chi_{w \neq 0} \mathcal{D}(C_{\phi,w}^*) \oplus \chi_{w=0} L^2(\mu)$ and $C_{\phi,w}^* f = C_{\phi,w}^* \big(\chi_{w \neq 0} f\big)$.

- $A = U|A|$ - the polar decomposition of A
  - U - a partial isometry satisfying $\mathcal{N}(A) = \mathcal{N}(U)$
  - $|A| = \sqrt{A^*A}$

Theorem

Suppose $C_{\phi,w}$ is densely defined and $C_{\phi,w} = U|C_{\phi,w}|$ is its polar decomposition. Then

(i) $|C_{\phi,w}| = M_{\sqrt{h_{\phi,w}}}$,

(ii) $U = C_{\phi,\widetilde{w}}$, where $\widetilde{w}\colon X \to \mathbb{C}$ is an $\mathscr{A}$-measurable function such that

$$\widetilde{w} = w \cdot \frac{1}{(h_{\phi,w} \circ \phi)^{1/2}} \text{ a.e. } [\mu].$$

(iii) $U^* f = h_{\phi,w}^{1/2} \mathsf{E}_{\phi,w}(f_w) \circ \phi^{-1}$, for $f \in L^2(\mu)$

(iv) $\mathcal{D}(|C^*_{\phi,w}|) = \{f \in L^2(\mu)\colon (h_{\phi,w} \circ \phi)^{1/2} \mathsf{E}_{\phi,w}(f_w) \in L^2(\mu)\}$,

(v) $|C^*_{\phi,w}| f = (h_{\phi,w} \circ \phi)^{1/2} \mathsf{E}_{\phi,w}(f_w)$ for $f \in \mathcal{D}(|C^*_{\phi,w}|)$.

- A is quasinormal iff $U|A| \subseteq |A|U$ (equivalently, $A^*AA = AA^*A$)

Theorem

Suppose $C_{\phi,w}$ is densely defined. Then the following are equivalent:

(i) $C_{\phi,w}$ is quasinormal,

(ii) $h_{\phi,w} \circ \phi = h_{\phi,w}$ a.e. $[\mu_w]$.

Theorem

Suppose C_ϕ is densely defined. Then the following are equivalent:

(i) C_ϕ is quasinormal,

(ii) $h_\phi \circ \phi = h$ a.e. $[\mu]$,

(iii) for every $n \in \mathbb{N}$, $h_{\phi^n} = h_\phi^n$ a.e. $[\mu]$.

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(iii) for every $n \in \mathbb{N}$, $h_{\phi^n} = h_\phi^n$ a.e. $[\mu]$.

- A is hyponormal iff $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|A^* f\| \leqslant \|Af\|$ for $f \in \mathcal{D}(A)$

Theorem

Suppose $C_{\phi,w}$ is densely defined. Then the following are equivalent:

(i) $C_{\phi,w}$ is hyponormal,

(ii) $h_{\phi,w} > 0$ a.e. $[\mu_w]$ and $\mathsf{E}_{\phi,w}\left(\frac{h_{\phi,w}\circ\phi}{h_{\phi,w}}\right) \leqslant 1$ a.e. $[\mu_w]$,

(ii) $h_{\phi,w} > 0$ a.e. $[\mu_w]$ and $\mathsf{E}_{\phi,w}\left(\frac{1}{h_{\phi,w}}\right) \leqslant \frac{1}{h_{\phi,w}\circ\phi}$ a.e. $[\mu_w]$.

- A is cohyponormal iff $\mathcal{D}(A^*) \subseteq \mathcal{D}(A)$ and $\|Af\| \leqslant \|A^*f\|$ for $f \in \mathcal{D}(A^*)$

Theorem

Suppose $C_{\phi,w}$ is densely defined. Then the following are equivalent:

(i) $C_{\phi,w}$ is cohyponormal,

(ii) the following conditions hold:

(ii-a) $h_{\phi,w} = 0$ on $\{w = 0\}$ a.e. $[\mu]$,

(ii-b) $\mathsf{E}_{\phi,w}(\theta^2|g|^2) \leqslant |\mathsf{E}_{\phi,w}(g)|^2$ a.e. $[\mu_w]$, for every $g \in L^2(\mu_w)$, where $\theta^2 = \frac{h_{\phi,w}}{h_{\phi,w}\circ\phi}$ a.e. $[\mu_w]$.

Theorem

Suppose $C_{\phi,w}$ is densely defined. Then the following are equivalent:

(i) $C_{\phi,w}$ is cohyponormal,

(ii) the following conditions hold:

(ii-a) $h_{\phi,w} = 0$ on $\{w = 0\}$ a.e. $[\mu]$,

(ii-b) $\chi_{\{h_{\phi,w}>0\}} L^2(\mu) \subseteq \mathcal{R}(\mathsf{E}_{\phi,w})$,

(ii-c) $h_{\phi,w} \leqslant h_{\phi,w} \circ \phi$ a.e. $[\mu_w]$.

Moreover, if $C_{\phi,w}$ is cohyponormal, then

(iii) $\mathsf{E}_{\phi,w}(h_{\phi,w}) = h_{\phi,w}$ a.e. $[\mu_w]$,

(iv) $M_\theta \in \mathbf{B}\big(L^2(\mu)\big)$, M_θ is a contraction, $\mathcal{R}(\mathsf{E}_{\phi,w})$ reduces M_θ and

$$M_\theta = M_\theta|_{\mathcal{R}(\mathsf{E}_{\phi,w})} \oplus 0|_{\mathcal{N}(\mathsf{E}_{\phi,w})}.$$

Theorem

Suppose $C_{\phi,w}$ is densely defined. Then the following are equivalent:

(i) $C_{\phi,w}$ is normal,

(ii) the following conditions hold:

(ii-a) $h_{\phi,w} = 0$ on $\{w = 0\}$ a.e. $[\mu]$,

(ii-b) $L^2(\mu_w) = \mathcal{R}(\mathrm{E}_{\phi,w})$,

(ii-c) $h_{\phi,w} = h_{\phi,w} \circ \phi$ a.e. $[\mu_w]$.

Moreover, if $C_{\phi,w}$ is normal, then $\{h_{\phi,w} > 0\} = \{w \neq 0\}$ a.e. $[\mu]$.

- A is formally normal iff $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|Af\| = \|A^* f\|$ for $f \in \mathcal{D}(A)$

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Moreover, if $C_{\phi,w}$ is normal, then $\{h_{\phi,w} > 0\} = \{w \neq 0\}$ a.e. $[\mu]$.

- A is formally normal iff $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|Af\| = \|A^* f\|$ for $f \in \mathcal{D}(A)$

Proposition

Suppose $C_{\phi,w}$ is formally normal. Then $C_{\phi,w}$ is normal.

- A is symmetric iff $A \subseteq A^*$
- A is selfadjoint iff $A = A^*$

Proposition

Suppose $C_{\phi,w}$ is densely defined. If $C_{\phi,w}$ is symmetric or positive, then $C_{\phi,w}$ is selfadjoint.

Theorem

Suppose $C_{\phi,w}$ is densely defined. Then the following are equivalent:.

(i) $C_{\phi,w}$ is selfadjoint,

(ii) the following conditions hold:

(ii-a) $h_{\phi,w} = (w \circ \phi)w$ a.e. $[\mu]$,

(ii-b) C_{ϕ^2} is well-defined in $L^2(\mu_w)$ and $C_{\phi^2} = I_{L^2(\mu_w)}$.

- A is symmetric iff $A \subseteq A^*$
- A is selfadjoint iff $A = A^*$

Proposition

Suppose $C_{\phi,w}$ is densely defined. If $C_{\phi,w}$ is symmetric or positive, then $C_{\phi,w}$ is selfadjoint.

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Theorem

The the following are equivalent:.

(i) $C_{\phi,w}$ is well-defined, selfadjoint, and positive,

(ii) the following conditions hold:

  (ii-a) $w \geqslant 0$ a.e. $[\mu]$,

  (ii-b) C_ϕ is well-defined in $L^2(\mu_w)$ and $C_\phi = I_{L^2(\mu_w)}$.

(iii) C_ϕ is well-defined, $C_{\phi,w} = M_w$, and $w \geqslant 0$ a.e. $[\mu]$.

THANK YOU