A Shimorin-type analytic model for left-invertible operators

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- Let $T \in \mathbf{B}(\mathcal{H})$. We say that T is left-invertible if there exists $S \in \mathbf{B}(\mathcal{H})$ such that $ST = I$.
- The Cauchy dual operator T' of a left-invertible operator $T \in \mathbf{B}(\mathcal{H})$ is defined by

$$T' := T(T^*T)^{-1}.$$

- An operator T is left-invertible if and only if there exists a constant $c > 0$ such that $T^*T \geqslant cI$.
- We call T analytic if $\mathcal{H}_\infty := \bigcap_{i=1}^{\infty} T^i\mathcal{H} = \{0\}$.

The classical Wold decomposition

Theorem (H. Wold 1938)

Let U be a isometry on Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ is the direct sum of two subspaces reducing U, $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_p$ such that $U|_{\mathcal{H}_u} \in \mathbf{B}(\mathcal{H}_u)$ is unitary and $U|_{\mathcal{H}_p} \in \mathbf{B}(\mathcal{H}_p)$ is unitarily equivalent to a unilateral shift. This decomposition is unique and the canonical subspaces are defined by

$$\mathcal{H}_u = \bigcap_{n=1}^{\infty} U^n \mathcal{H} \quad \text{and} \quad \mathcal{H}_p = \bigoplus_{n=1}^{\infty} U^n E,$$

where $E := \mathcal{N}(U^*) = \mathcal{H} \ominus U\mathcal{H}$.

We shall say that an operator $T \in \mathbf{B}(\mathcal{H})$ admits Wold-type decomposition, if the subspaces $\mathcal{H}_\infty := \bigcap_{n=1}^{\infty} T^n\mathcal{H}$ and $E := \mathcal{N}(T^*)$ have the following properties:

- $\mathcal{H}_\infty$ is reducing for T and T is unitary on $\mathcal{H}_\infty$,
- $\mathcal{H} = \mathcal{H}_\infty \oplus [E]_T$,

where $[E]_T := \bigvee\{T^n x\colon x \in \mathcal{H}, n \in \mathbb{N}\}$.

Theorem (S. Shimorin 2001)

Assume that $T \in \mathbf{B}(\mathcal{H})$ satisfies one of the following conditions:

- $\|T^2x\| + \|x\|^2 \leq 2\|Tx\|^2$ *for* $x \in \mathcal{H}$,
- $\|T(x+y)\|^2 \leq 2(\|x\|^2 + \|Ty\|^2)$ *for* $x, y \in \mathcal{H}$

Then T admits Wold-type decomposition.

The construction of the Shimorin's model for a left-invertible analytic operator (S. Shimorin 2001)

- Let $T \in \mathbf{B}(\mathcal{H})$ be a left-invertible analytic operator and $E := \mathcal{N}(T^*)$.
- For evey $x \in \mathcal{H}$ define a vector-valued holomorphic functions U_x as

$$U_x(z) = \sum_{n=0}^{\infty} (P_E T'^{*n} x) z^n, \quad z \in \mathbb{D}(r(T')^{-1}),$$

  where T' is the Cauchy dual of T.
- The operator $U : \mathcal{H} \ni x \to U_x \in \mathscr{H}$ is injective.
- We equip the obtained space of analytic functions $\mathscr{H} := \{U_x : x \in \mathcal{H}\}$ with the inner product induced by $\mathcal{H}$.
- The operator $U : \mathcal{H} \ni x \to U_x \in \mathscr{H}$ becomes a unitary isomorphism.

The space $\mathscr{H}$ is a reproducing kernel Hilbert space in the following sense: *the reproducing kernel* for $\mathscr{H}$ is an $\mathbf{B}(E)$-valued function of two variables $\kappa_{\mathscr{H}} : \Omega \times \Omega \to \mathbf{B}(E)$ such that

- for any $e \in E$ and $\lambda \in \Omega$

$$\kappa_{\mathscr{H}}(\cdot, \lambda)e \in \mathscr{H},$$

- for any $e \in E$, $f \in \mathscr{H}$ and $\lambda \in \Omega$

$$\langle f(\lambda), e\rangle_E = \langle f, \kappa_{\mathscr{H}}(\cdot, \lambda)e\rangle_{\mathscr{H}}.$$

Theorem (S.Shimorin 2001)

The space $\mathscr{H}$ is a reproducing kernel Hilbert space and the reproducing kernel $\kappa_{\mathscr{H}} : \mathbb{D}(r(T')^{-1}) \times \mathbb{D}(r(T')^{-1}) \to \mathbf{B}(E)$ is given by

$$\kappa_{\mathscr{H}}(z, \lambda) = P_E(I - \lambda T'^*)^{-1}(I - zT')^{-1}|_E.$$

Theorem (S. Shimorin 2001)

Let $T \in \mathbf{B}(\mathcal{H})$ be a left-invertible analytic operator. Then the operator T is unitarily equivalent to the operator $\mathscr{M}_z$ of multiplication by z on $\mathscr{H}$ and T'^* is unitarily equivalent to the operator $\mathscr{L}$ given by

$$(\mathscr{L}f)(z) = \frac{f(z) - f(0)}{z}, \quad f \in \mathscr{H}.$$

The construction of the Shimorin-type analytic model for a left-invertible operator (P.P. 2018)

- Let $T \in \mathbf{B}(\mathcal{H})$ be a left-invertible operator.
- Let E be a subspace of $\mathcal{H}$ denote by $[E]_{T^*,T'}$ the following subspace of $\mathcal{H}$:

$$[E]_{T^*,T'} := \bigvee \left(\{T^{*n}x \colon x \in E, n \in \mathbb{N}\} \cup \{T'^{n}x \colon x \in E, n \in \mathbb{N}\}\right),$$

- We choose closed subspace E such that $[E]_{T^*,T'} = \mathcal{H}$. where T' is the Cauchy dual of T.
- For each $x \in \mathcal{H}$, define a formal Laurent series U_x with vector coefficients as

$$U_x(z) = \sum_{n=1}^{\infty}(P_E T^n x)\frac{1}{z^n} + \sum_{n=0}^{\infty}(P_E T'^{*n} x)z^n,$$

where T' is the Cauchy dual of T.

- The operator $U : \mathcal{H} \ni x \to U_x \in \mathscr{H}$ is injective.
- We equip the obtained space of formal Laurent series $\mathscr{H} := \{U_x : x \in \mathcal{H}\}$ with the inner product induced by $\mathcal{H}$.
- The operator $U : \mathcal{H} \ni x \to U_x \in \mathscr{H}$ becomes a unitary isomorphism.

Theorem (P.P. 2018)

Let $T \in \mathbf{B}(\mathcal{H})$ be a left-invertible operator and E be a closed subspace of $\mathcal{H}$ such that $[E]_{T^*,T'} = \mathcal{H}$. Then the operator T is unitary equivalent to the operator $\mathscr{M}_z : \mathscr{H} \to \mathscr{H}$ of multiplication by z on $\mathscr{H}$ given by

$$(\mathscr{M}_z f)(z) = zf(z), \quad f \in \mathscr{H},$$

and operator T'^* is unitary equivalent to the operator $\mathscr{L} : \mathscr{H} \to \mathscr{H}$ given by

$$(\mathscr{L} f)(z) = \frac{f(z) - (P_{\mathcal{N}(\mathscr{M}_z^*)} f)(z)}{z}, \quad f \in \mathscr{H}.$$

Theorem (P.P. 2018)

Let $T \in \mathbf{B}(\mathcal{H})$ be left-invertible and analytic, $\mathscr{H}_1$, U_1 be the Hilbert space and the unitary map construceted in our analytic model with $E := \mathcal{N}(T^*)$ and $\mathscr{H}_2$, U_2 be the Hilbert space and the unitary map obtained in Shimorin's construction. Then $\mathscr{H}_1 = \mathscr{H}_2$ and $U_1 = U_2$.

Theorem (P.P. 2018)

Let $T \in \mathbf{B}(\mathcal{H})$ be a left-invertible operator and E be a closed subspace of $\mathcal{H}$ such that $[E]_{T^*,T'} = \mathcal{H}$. Then for every $m \in \mathbb{N}$ the following assertions hold:

(i) *$T'^m E$ is a closed supspace and $[T'^m E]_{T^*,T'} = \mathcal{H}$,*

(ii) *the mapping $\Phi_m : \mathscr{H}_0 \to \mathscr{H}_m$ defined by*

$$\Phi_m\Big(\sum_{n=-\infty}^{\infty} a_n z^n\Big) = \sum_{n=-\infty}^{\infty} (V_m a_{m+n}) z^n, \quad \sum_{n=-\infty}^{\infty} a_n z^n \in \mathscr{H}_0$$

is a unitary isomorphism, where $\mathscr{H}_k$ for $k \in \mathbb{N}$ is the Hilbert space constructed in our analytic model with subspace $T'^k E$ and $V_k : E \to T'^k E$ for $k \in \mathbb{N}$ is defined by,

$$Ve = P_{T'^k E} T^k e, \qquad e \in E.$$

Theorem (P.P. 2018)

Let $T \in \mathbf{B}(\mathcal{H})$ be a left-invertible operator and E be a closed subspace of $\mathcal{H}$ such that $[E]_{T^*,T'} = \mathcal{H}$. Let

$$r^+ := \liminf_{n\to\infty} \left\| P_E {T'^*}^n \right\|^{-\frac{1}{n}},$$

$$r^- := \limsup_{n\to\infty} \left\| P_E T^n \right\|^{\frac{1}{n}}.$$

If $r^+ > r^-$, then formal Laurent series

$$U_x(z) = \sum_{n=1}^{\infty} (P_E T^n x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E {T'^*}^n x) z^n,$$

represent analytic functions on annulus $\mathbb{A}(r^-, r^+)$.

Theorem (P.P. 2018)

Let $T \in \mathbf{B}(\mathcal{H})$ be a left-invertible operator and E be a closed subspace of $\mathcal{H}$ such that $[E]_{T^*,T'} = \mathcal{H}$ and the series

$$U_x(z) = \sum_{n=1}^{\infty} (P_E T^n x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E {T'^*}^n x) z^n,$$

is convergent in E on an annulus $\mathbb{A}(r^-, r^+)$ with $r^- < r^+$ and $r^-, r^+ \in [0, \infty)$ for every $x \in \mathcal{H}$. Then $\mathscr{H}$ is a reproducing kernel Hilbert space of E-valued holomorphic functions on $\mathbb{A}(r^-, r^+)$. The reproducing kernel $\kappa_{\mathscr{H}} : \mathbb{A}(r^-, r^+) \times \mathbb{A}(r^-, r^+) \to \mathbf{B}(E)$ associated with $\mathscr{H}$ is given by

$$\begin{aligned}\kappa_{\mathscr{H}}(z, \lambda) = & \sum_{i,j \geqslant 1} P_E T^i T^{*j}|_E \frac{1}{z^i} \frac{1}{\bar{\lambda}^j} + \sum_{i \geqslant 1, j \geqslant 0} P_E T^i T'^j|_E \frac{1}{z^i} \bar{\lambda}^j \\ & + \sum_{i \geqslant 0, j \geqslant 1} P_E {T'^*}^i T^{*j}|_E z^i \frac{1}{\bar{\lambda}^j} + \sum_{i,j \geqslant 0} P_E {T'^*}^i T'^j|_E z^i \bar{\lambda}^j.\end{aligned} \tag{1}$$

Theorem (P.P. 2018)

Moreover, the following assertions hold.

(i) *For any* $\lambda \in \mathbb{A}(r^-, r^+)$

$$\sum_{n=1}^{\infty}(P_E T^n)\frac{1}{\lambda^n} + \sum_{n=0}^{\infty}(P_E T^{*\prime n})\lambda^n \in \mathbf{B}(\mathcal{H}, E), \qquad (2)$$

$$\sum_{n=1}^{\infty} T^{*n}\frac{1}{\lambda^n} + \sum_{n=0}^{\infty} T'^n\lambda^n \in \mathbf{B}(E, \mathcal{H}), \qquad (3)$$

(ii) *The series* (1), (2) *and* (3) *converges absolutely and uniformly in operator norm on any compact set contained in* $\mathbb{A}(r^-, r^+) \times \mathbb{A}(r^-, r^+)$, $\mathbb{A}(r^-, r^+)$ *and* $\mathbb{A}(r^-, r^+)$, *respectively.*

(iii) *the function* $\mathbb{A}(r^-, r^+) \ni \lambda \to \kappa_{\mathscr{H}}(\cdot, \bar{\lambda})e \in \mathscr{H}$, $e \in E$ *is holomorphic and given by*

$$\kappa_{\mathscr{H}}(\cdot, \bar{\lambda})e = \sum_{n=1}^{\infty} UT^{*n}e\frac{1}{\lambda^n} + \sum_{n=0}^{\infty} UT'^n e\lambda^n, \qquad \lambda \in \mathbb{A}(r^-, r^+).$$

Theorem (P.P. 2018)

Let $T \in \mathbf{B}(\mathcal{H})$ be a left-invertible operator and E be a closed subspace of $\mathcal{H}$ such that $[E]_{T^*,T'} = \mathcal{H}$ and the series

$$U_x(z) = \sum_{n=1}^{\infty} (P_E T^n x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E {T'^*}^n x) z^n,$$

convergent in E for every $x \in \mathcal{H}$ on open nonempty subset $\Omega \subset \mathbb{C}$. Then the following assertions hold:

(i) *the point spectrum of T is empty, that is $\sigma_p(T) = \emptyset$,*

(ii) $\mathscr{M}_z^* \kappa_{\mathscr{H}}(\cdot, \mu) g = \bar{\mu} \kappa_{\mathscr{H}}(\cdot, \mu) g$, *for every $\mu \in \Omega$, $g \in E$,*

(iii) $\bar{\Omega} \subset \sigma_p(T^*)$,

(iv) $\bigvee \{\mathcal{N}(T^* - \bar{\mu})\colon \mu \in U\} = \mathcal{H}$, *where $U \subset \Omega$ and $\operatorname{int} U \neq \emptyset$.*

Composition operators in L^2-spaces

- $(X, \mathcal{A}, \mu)$ is a σ-finite measure space
- $\phi : X \to X$ is an $\mathcal{A}$-measurable transformation, i.e., $\phi^{-1}(\Delta) \in \mathcal{A}$ for every $\Delta \in \mathcal{A}$
- If the measure $\mu \circ \phi^{-1}$ given by $\mu \circ \phi^{-1}(\Delta) = \mu(\phi^{-1}(\Delta))$ for $\Delta \in \mathcal{A}$ is absolutely continuos with respect to μ (we say that μ is **nonsingular**), then the operator C_ϕ in $L^2(\mu)$ given by
  $\mathcal{D}(C_\phi) = \{f \in L^2(\mu) : f \circ \phi \in L^2(\mu)\}$,
  $C_\phi f = f \circ \phi, f \in \mathcal{D}(C_\phi)$
  is well-defined
- We call it a **composition** operator with **symbol** ϕ

- For $x \in X$ the set

$$[x]_\phi = \{y \in X : \text{there exist } i, j \in \mathbb{N} \text{ such that } \phi^{(i)}(x) = \phi^{(j)}(y)\}$$

is called the orbit of f containing x.

- If $x \in X$ and $\phi^{(i)}(x) = x$ for some $i \in \mathbb{Z}_+$ then the cycle of ϕ containing x is the set

$$\mathscr{C}_\varphi = \{\phi^{(i)}(x) \colon i \in \mathbb{N}\}$$

- We will only consider composition functions with one orbit, since an orbit induces a reducing subspace to which the restriction of the weighted composition operator is again a weighted composition operator.
- Any self-map $\phi : X \to X$ induces a directed graph (X, E^ϕ) given by

$$E^\phi = \{(x, y) \in X \times X \colon x = \phi(y)\}$$

Lemma (P.P. 2018)

Let X be a countable set, $w : X \to \mathbb{C}$ be a complex function on X and $\varphi : X \to X$ be a transformation of X. Let $C_{\varphi,w}$ be a weighted composition operator in $\ell^2(X)$ and

$$E := \begin{cases} \bigoplus_{x \in \mathrm{Gen}_\varphi(1,1)} \langle e_x \rangle \oplus \mathcal{N}((C_{\varphi,w}|_{\ell^2(\mathrm{Des}(x))})^*) & \text{if } \varphi \text{ has a cycle,} \\ \langle e_\omega \rangle \oplus \mathcal{N}(C^*_{\varphi,w}) & \text{otherwise,} \end{cases} \tag{4}$$

where $\mathrm{Des}(x) := \bigcup_{n=0}^{\infty} \varphi^{(-n)}(x)$ and ω is a generalized root of the tree. Then the subspace E has the following properties:

(i) $[E]_{C^*_{\varphi,w}, C_{\varphi,w'}} = \mathcal{H}$ *and* $[E]_{C_{\varphi,w}, C^*_{\varphi,w'}} = \mathcal{H}$,

(ii) $E \perp C^n_{\varphi,w} E$ *and* $E \perp C^n_{\varphi,w'} E$, $n \in \mathbb{Z}_+$.

The non-negative number

$$r^{+}_{w,\varphi} := \liminf_{n\to\infty} \Big(\sum_{\substack{x\in W_n^{E,\varphi}\\ n\geqslant 0}} |w'(x)w'(\varphi(x))\cdots w'(\varphi^{(n-1)}(x))|^2 \Big)^{-\frac{1}{2n}} \quad (5)$$

will be called the outer radius of convergence for $C_{\varphi,w}$, and similarly the non-negative number

$$r^{-}_{w,\varphi} := \begin{cases} \sqrt[\tau]{\prod_{x\in\mathscr{C}_\varphi} |w(x)|} & \text{if } \varphi \text{ has a cycle,} \\ \limsup_{n\to\infty} \sqrt[n]{|w(\varphi^1(\omega))w(\varphi^2(\omega))\dots w(\varphi^n(\omega))|} & \text{otherwise,} \end{cases} \quad (6)$$

where $\tau := \operatorname{card} \mathscr{C}_\varphi$ will be called the inner radius of convergence for $C_{\varphi,w}$.

Theorem (P.P. 2018)

Let X be a countable set, $w : X \to \mathbb{C}$ be a complex function on X and $\varphi : X \to X$ be a transformation of X, which has finite branching index. Let $C_{\varphi,w}$ be a left-invertible weighted composition operator in $\ell^2(X)$. If $r^+_{w,\varphi} > r^-_{w,\varphi}$, then there exist a z-invariant reproducing kernel Hilbert space $\mathscr{H}$ of E-valued holomorphic functions defined on the annulus $\mathbb{A}(r^-_{w,\varphi}, r^+_{w,\varphi})$ and a unitary mapping $U : \ell^2(X) \to \mathscr{H}$ such that $\mathscr{M}_z U = UC_{\varphi,w}$, where $\mathscr{M}_z$ denotes the operator of multiplication by z on $\mathscr{H}$, where E is as in (4).

Theorem (P.P. 2018)

Moreover, the following assertions hold :

(i) *the reproducing kernel* $\kappa_{\mathscr{H}} : \mathbb{A}(r^-_{w,\varphi}, r^+_{w,\varphi}) \times \mathbb{A}(r^-_{w,\varphi}, r^+_{w,\varphi}) \to \mathbf{B}(E)$ *associated with* $\mathscr{H}$ *has the property that* $\kappa_{\mathscr{H}}(\cdot, w)g \in \mathscr{H}$ *and* $\langle Uf, \kappa_{\mathscr{H}}(\cdot, w)g\rangle = \langle (Uf)(w), g\rangle$ *for* $f, g \in \ell^2(X)$.

(ii) *the reproducing kernel* $\kappa_{\mathscr{H}}$ *has the following form:*

$$\kappa_{\mathscr{H}}(z, \lambda) = \sum_{i,j\geqslant 1} A_{i,j}\frac{1}{z^i}\frac{1}{\lambda^j} + \sum_{i\geqslant 1, j\geqslant 0} B_{i,j}\frac{1}{z^i}\lambda^j + \sum_{i\geqslant 0, j\geqslant 1} C_{i,j}z^i\frac{1}{\lambda^j} + \sum_{i,j\geqslant 0} D_{i,j}z^i\lambda^j,$$

where $A_{i,j}, B_{i,j}, C_{i,j}, D_{i,j} \in \mathbf{B}(E)$;

Theorem (P.P. 2018)

(iii) *if φ does not have a cycle, then the linear subspace generated by E-valued polynomials in z and $\tilde{E}$-valued polynomials involving only negative powers of z is dense in $\mathscr{H}$, that is*

$$\bigvee\left(\{z^n E\colon n \in \mathbb{N}\} \cup \{\frac{1}{z^n}\tilde{E}\colon n \in \mathbb{Z}_+\}\right) = \mathscr{H},$$

where $\tilde{E} := \bigvee\{e_x\colon x \in \mathrm{Gen}_\varphi(1,1)\}$;

Theorem (P.P. 2018)

if φ has a cycle $\mathscr{C}_\varphi$ with $\tau := \operatorname{card} \mathscr{C}_\varphi$, then there exist τ functions $f_1, \ldots, f_\tau$ on $\mathbb{A}(r^-_{w,\varphi}, r^+_{w,\varphi})$ given by the following Laurent series

$$f_i(z) := \sum_{k=0}^{\infty} \sum_{i=1}^{\tau} \Lambda^k A_i \frac{1}{z^{k\tau+i}}, \quad i = 1, ..., \tau,$$

where $\Lambda := \prod_{x \in \mathscr{C}_\varphi} w(x)$ such that the linear subspace generated by E-valued polynomials in z and the above functions is dense in $\mathscr{H}$, that is

$$\bigvee(\{z^n E \colon n \in \mathbb{N}\} \cup \{f_i \colon i \in \{1, \ldots \tau\}\}) = \mathscr{H}.$$

Example (P.P. 2018)

Fix $m \in \mathbb{N}$ and set $X = \{0, 1, \ldots, m\}$. Let $w : X \to \mathbb{C}$ be a function and define a mapping $\varphi : X \to X$ by

$$\varphi(i) = \begin{cases} i+1 & \text{if } i < m \\ 0 & \text{if } i = m \end{cases}$$

(see Figure 1). Set $\Lambda := w(0)w(1)\ldots w(m)$. Let $C_{\varphi,w}$ be the left-invertible composition operator in $\mathbb{C}^{m+1}$.

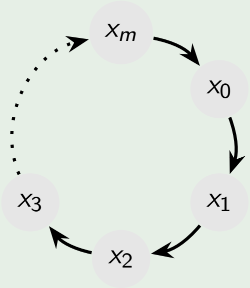

Figure:

Example

Let $E := \text{lin}\{e_1\}$. It is easy to see that $[E]_{S_\lambda, S'^*_\lambda} = \mathcal{H}$. One can o verify that

$$P_E C_{\varphi,w}^{mk+r} x = \Lambda^k \Big(\prod_{i=0}^{r-1} w(i) \Big) x_r e_0,$$

$$P_E C_{\varphi,w}^{\prime *(mk+r)} x = \frac{1}{\Lambda^k} \Big(\prod_{i=m+1-r}^{m} w(i) \Big)^{-1} x_{n+1-r} e_0,$$

for $r < n$, $r, k \in \mathbb{N}$.

Example (P.P. 2018)

This shows that formal Laurent series takes the following form:

$$U_x(z) = \sum_{k=1}^{\infty}\sum_{r=0}^{n-1}\Big(\Lambda^k\Big(\prod_{i=0}^{r-1} w(i)\Big)x_r e_0\Big)\frac{1}{z^{nk+r}}$$
$$+\sum_{k=0}^{\infty}\Big(\sum_{r=0}^{n-1}\frac{1}{\Lambda^k}\Big(\prod_{i=m+1-r}^{m} w(i)\Big)^{-1} x_{n+1-r}e_0\Big)z^{nk+r}.$$

Since $C^*_{\varphi,w}$ acts on the finite dimensional space, the spectrum of $C^*_{\varphi,w}$ is finite. Therefore, by assertion (iii) of Theorem 8 the above series does not converge absolutely on any open subset of $\mathbb{C}$. Alternatively, one can prove this fact directly by calculating convergences radii.

Example (Bilateral weighted shift)

Let $S_\lambda : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be a bilateral weighted shift with weights $\{\lambda_n\}_{n\in\mathbb{Z}}$ and $\{e_n\}_{n\in\mathbb{Z}}$ be the standard orthonormal basis of $\ell^2(\mathbb{Z})$. Then

$$S_\lambda e_n = \lambda_{n+1} e_{n+1}, \qquad n \in \mathbb{Z}$$

Let $E := \operatorname{lin}\{e_0\}$. It is easy to see that $[E]_{S_\lambda^*, S_\lambda'} = \mathcal{H}$. It is worth noting that $\mathcal{N}(S_\lambda^*) = \{0\}$ and thus $[\mathcal{N}(S_\lambda^*)]_{S_\lambda^*, S_\lambda'} = \{0\}$. This phenomenon is quite different comparing with the case of left-invertible and analytic operators in which $[\mathcal{N}(T^*)]_{T^*, T'} = \mathcal{H}$, where T is in this class.

Example

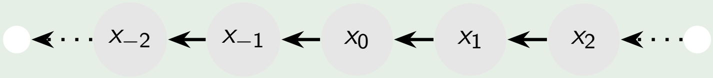

Figure:

It is a matter of routine to verify that the Cauchy dual $S_\lambda'^*$ of S_λ has the following form

$$S_\lambda'^* e_n = \frac{1}{\lambda_n} e_{n-1}, \qquad n \in \mathbb{Z}.$$

It is now easily seen that

$$P_E (S_\lambda'^*)^n x = \Big(\prod_{i=1}^{n} \lambda_i\Big)^{-1} x_n e_0, \qquad n \in \mathbb{Z}_+,$$

Example

and

$$P_E S_\lambda^n x = \Big(\prod_{i=-n+1}^{0} \lambda_i \Big) x_{-n} e_0, \qquad n \in \mathbb{Z}_+.$$

Therefore, the formal Laurent series takes the form

$$U_x(z) = \sum_{n=1}^{\infty} \Big(\prod_{i=-n+1}^{0} \lambda_i \Big) x_{-n} \frac{1}{z^n} + \sum_{n=0}^{\infty} \Big(\prod_{i=1}^{n} \lambda_i \Big)^{-1} x_n z^n.$$

One can show that

$$r_{w,\varphi}^{+} = \liminf_{n\to\infty} \sqrt[n]{\prod_{i=1}^{n} |\lambda_i|}$$

Example

and

$$r^{-}_{w,\varphi} = \limsup_{n\to\infty} \sqrt[n]{\prod_{i=-n+1}^{0} |\lambda_i|}.$$

In this case, the reproducing kernel $\kappa_{\mathscr{H}} : \mathbb{A}(r^-, r^+) \times \mathbb{A}(r^-, r^+) \to \mathbf{B}(E)$ is given by

$$\kappa_{\mathscr{H}}(z,\lambda) = \sum_{i=1}^{\infty} \prod_{i=-n+1}^{0} |\lambda_i|^2 \frac{1}{(z\bar{\lambda})^i} + \sum_{i=0}^{\infty} \Big(\prod_{i=1}^{n} |\lambda_i|^2\Big)^{-1} (z\bar{\lambda})^i.$$

Example

Set $m \in \mathbb{N}$ and $X = \{0, 1, \dots m\} \sqcup \{(0, i) \colon i \in \mathbb{N}\}$. Let $w : X \to \mathbb{C}$ be a measurable function and $\varphi : X \to X$ be transformation of X defined by

$$\varphi(x) = \begin{cases} (0, i-1) & \text{for } x = (0,i),\ i \in \mathbb{N} \setminus \{0\}, \\ m & \text{for } x = (0,0), \\ i-1 & \text{for } x = i \text{ and } i \in \{1, \dots, m\}, \\ m & \text{for } x = 0, \end{cases}$$

(see Figure 3). Let $C_{\varphi,w} : \ell^2(X) \to \ell^2(X)$ be a left-invertible composition operator. It is easily seen that

$$C_{\varphi,w} e_x = \begin{cases} w((0,i+1)) e_{(0,i+1)} & \text{for } x = (0,i),\ i \in \mathbb{N} \setminus \{0\} \\ w(i+1) e_{i+1} & \text{for } x = i \text{ and } i \in \{0, 1, \dots, m\} \\ w(0) e_0 + w((0,0)) e_{(0,0)} & \text{for } x = m. \end{cases}$$

Example

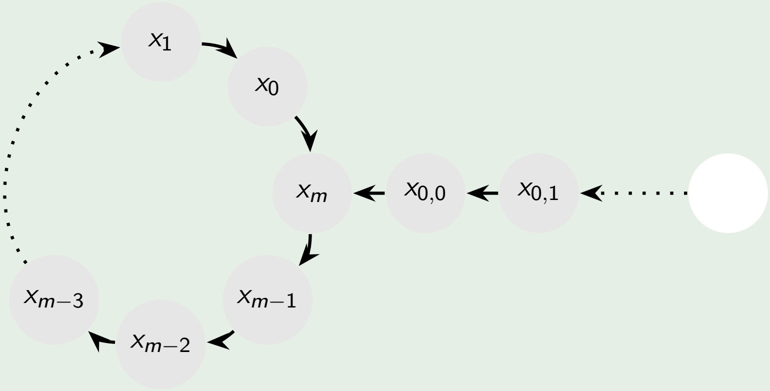

Figure:

It is routine to verify that $\mathcal{N}(C^*_{\varphi,w}) = \text{lin}\{\overline{w((0,0))}e_0 - \overline{w(0)}e_{(0,0)}\}$. Let $E := \text{lin}\{e_{(0,0)}\}$. One can check that this one-dimensional subspace satisfies $[E]_{T^*,T'} = \mathcal{H}$.

Example

This implies that

$$P_E(C_{\varphi,w'}^*)^n x = \Big(\prod_{i=1}^{n} w(0,i)\Big)^{-1} x_n e_{(0,0)},$$

$$P_E C_{\varphi,w}^{nm+r+1} x = \Lambda^n w((0,0))\Big(\prod_{i=0}^{r-1} w(m-i)\Big) x_{m-r} e_{(0,0)},$$

for $r < m$, $r, n \in \mathbb{N}$. Hence, the Hilbert space $\mathscr{H}$ consist of the functions of the form

$$U_x(z) = \sum_{n=1}^{\infty}\sum_{r=0}^{k}\Lambda^k w((0,0))\Big(\prod_{i=0}^{r-1} w(m-i)\Big) x_{m-r}\frac{1}{z^{nm+r+1}} \qquad (7)$$

$$+ \sum_{n=0}^{\infty}\Big(\prod_{i=1}^{n} w((0,i))\Big)^{-1} x_n z^n.$$

Example

The formulas for the inner and outer radius of convergence take the following form

$$r^{+}_{w,\varphi} = \liminf_{n\to\infty} \sqrt[n]{\prod_{i=1}^{n} |w((0,i))|}$$

and

$$r^{-}_{w,\varphi} = \sqrt[m+1]{\prod_{i=0}^{m} |w(i)|}.$$

Example

The reproducing kernel $\kappa_{\mathscr{H}} : \mathbb{A}(r^-, r^+) \times \mathbb{A}(r^-, r^+) \to \mathbf{B}(E)$ takes the form

$$\kappa_{\mathscr{H}}(z, \lambda) = \sum_{i \geqslant 1, j \geqslant 1} \Lambda^i \bar{\Lambda}^j |w((1,0))|^2 \Big(\prod_{i=0}^{r-1} |w(m-i)|^2 \Big) \frac{1}{z^{im+r+1} \bar{\lambda}^{jm+r+1}} + \sum_{i=0}^{\infty} \Big(\prod_{i=1}^{n} |w((0,i))|^2 \Big)^{-1} (z\bar{\lambda})^i.$$

Thank you for your attention!