q-deformed Araki-Woods von Neumann algebra

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von Neumann Algebra

- Let $\mathcal{H}$ be a separable Hilbert space. A unital $*$-subalgebra $M \subseteq \mathcal{B}(\mathcal{H})$ is von Neumann algebra (shortly vNa) if M is WOT closed ($\Leftrightarrow$ SOT closed $\Leftrightarrow M'' = M$).

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- **Tomita-Takesaki modular Theory:** Let ϕ be a f.n state on M and $L^2(M, \phi)$ be the GNS the Hilbert space.
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- $S_\phi = J_\phi \Delta_\phi$ (the polar decomposition of S_ϕ) and $\sigma_t^\phi(\cdot) = \Delta_\phi^{it}(\cdot)\Delta_\phi^{-it}$.
  1. J_ϕ and Δ_ϕ are called modular conjugation and modular operator resp.
  2. $J_\phi M J_\phi = M'$ and $\forall t \in \mathbb{R}$, $\sigma_t^\phi(M) = M$.
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- A vNa algebra $M \subseteq \mathcal{B}(\mathcal{H})$ is called factor if $M \cap M' = \mathbb{C}1$.

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  1. $III_\lambda, \lambda \in (0,1)$, if $S(M) = \{0\} \cup \{\lambda^n; n \in \mathbb{Z}\}$
  2. III_0, if $S(M) = \{0,1\}$
  3. III_1, if $S(M) = [0,\infty)$

Construction of the q-Fock space

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- Define
$$\langle \xi, \eta \rangle_U = \langle \frac{2}{1 + A^{-1}} \xi, \eta \rangle_{\mathcal{H}_{\mathbb{C}}}, \quad \xi, \eta \in \mathcal{H}_{\mathbb{C}}.$$

  Let $\mathcal{H} = \overline{\mathcal{H}_{\mathbb{C}}}^{\|\cdot\|}$ and $(\mathcal{H}_{\mathbb{R}}, \|\cdot\|) \ni \xi \overset{\imath}{\mapsto} \xi \in (\mathcal{H}, \|\cdot\|_U)$, is an isometric embedding.

Construction of the q-Fock space

- Let $-1 < q < 1$. Then q-Fock space $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$ with respect to the inner product;

$$\begin{aligned}\langle \xi_1 \otimes \cdots \otimes \xi_n, \zeta_1 \otimes \cdots \otimes \zeta_m \rangle_q \\ = \delta_{m,n} \sum_{\pi \in S_n} q^{i(\pi)} \langle \xi_1, \zeta_{\pi(1)} \rangle_U \cdots \langle \xi_n, \ \zeta_{\pi(n)} \rangle_U,\end{aligned}$$

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- In particular, $\mathcal{F}_0(\mathcal{H})$ is the usual Fock space $\mathcal{F}(\mathcal{H})$

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- **q-commutation relation**
  $$c_q(\xi)^* c_q(\eta) - q c_q(\eta) c_q(\xi)^* = \langle \xi, \eta \rangle, \ \forall \xi, \eta \in \mathcal{H}_{\mathbb{C}}.$$

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- $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ is known as the *q-deformed Araki-Woods von Neumann algebra* constructed by Hiai and $\varphi := \langle\Omega, \cdot\Omega\rangle_q$ called the *q-quasi free state*, is a f.n state of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ and $\mathcal{F}_q(\mathcal{H})$ is the GNS Hilbert space of $\Gamma_q(\mathcal{H}_\mathbb{R}, U_t)''$ associated to φ.

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- $\Gamma_q(H_{\mathbb{R}}, U_t)''$ are one of the interesting deformation of Voiculescu's free Gaussian functor, i.e, when $q = 0$ and $U_t = id$, $\Gamma_0(\mathcal{H}_{\mathbb{R}}, id_t)''$ is the Voiculescu free Gaussian functor and it is isomorphic to $L(\mathbb{F}_{\dim(\mathcal{H}_{\mathbb{R}})})$. It got three deformation.

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- Case 1: $\mathbf{q \neq 0}$ and $\mathbf{U_t = id}$.
  It is called Bozeko and Speicher von Neumann algebras. For $dim(\mathcal{H}_{\mathbb{R}}) \geq 2$, the q-Gaussian von Neumann algebras $\Gamma_q(\mathcal{H}_{\mathbb{R}})$ are non-injective, solid, strongly solid, non Γ factors with w^*-completely contractive approximation property. Further, $\Gamma_q(\mathcal{H}_{\mathbb{R}}) \cong L(\mathbb{F}_{dim(\mathcal{H}_{\mathbb{R}})})$ for values of q sufficiently close to zero.

- Case 2: $\mathbf{q} = \mathbf{0}$ and $\mathbf{U_t} \neq \mathbf{id}$.
  This is the Shlyakhtenko functor. These von Neumann algebras are type III counterparts of the free group factors. In short, they satisfy the complete metric approximation property, lack Cartan subalgebras, are strongly solid, and, they satisfy Connes' bicentralizer problem when they are type III_1.

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  This is the Shlyakhtenko functor. These von Neumann algebras are type III counterparts of the free group factors. In short, they satisfy the complete metric approximation property, lack Cartan subalgebras, are strongly solid, and, they satisfy Connes' bicentralizer problem when they are type III_1.
- Case 3: $\mathbf{q} \neq \mathbf{0}$ and $\mathbf{U_t} \neq \mathbf{id}$.
  This is the q-deformed functor due to Hiai for $-1 < q < 1$. Hiai's functor is the main topic of this paper. It is a combination of Bożejko-Speicher's functor and Shlyakhtenko's functor.

- **Centralizer:**

$$(\Gamma(H_{\mathbb{R}}, U_t)'')^{\varphi} = \{x \in \Gamma(H_{\mathbb{R}}, U_t)'' : \quad \sigma_t^{\varphi}(x) = x, \quad \forall t \in \mathbb{R}\}$$

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- **Bicentralizer:**

$$B_{\psi} = \{y \in M : \lim_{n\to\infty} |\rho([y, x_n])| = 0 \ , \forall \rho \in M_*, \\ \forall (x_n) \text{ with } \lim_{n\to\infty} \|(x_n\psi - \psi x_n)\| = 0\}.$$

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Theorem (with K. Mukherjee)

Let $\xi \in \mathcal{H}_\mathbb{R}$ with $\|\xi\|_U = 1$. Then;

1. *$\exists!$ φ-preserving f.n conditional expectation $\mathbb{E}_\xi : M_{q,U} \to M_\xi$ if and only if $s_q(\xi) \in M_{q,U}^{\varphi}$, equivalently $U_t\xi = \xi$ for all $t \in \mathbb{R}$.*

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2. *If $U_t\xi = \xi$ for all $t \in \mathbb{R}$ and $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$. Then M_ξ is a φ-strongly mixing diffuse masa in $M_{q,U}$ whose distribution obeys semicircular law and left-right measure is Lebesgue absolutely continuous. In particular, M_ξ is singular masa in $M_{q,U}$.*

Centralizer and factoriality

- Let $t \mapsto U_t$ be a strongly orthogonal representation of $\mathbb{R}$ on the real Hilbert space $\mathcal{H}_{\mathbb{R}}$ as before. Then one can decompose as;
- $$(\mathcal{H}_{\mathbb{R}}, U_t) = \left(\bigoplus_{j=1}^{N_1}(\mathbb{R}, \mathrm{id})\right) \oplus \left(\bigoplus_{k=1}^{N_2}(\mathcal{H}_{\mathbb{R}}(k), U_t(k))\right) \oplus (\widetilde{\mathcal{H}}_{\mathbb{R}}, \widetilde{U}_t),$$
- where
  $$\mathcal{H}_{\mathbb{R}}(k) = \mathbb{R}^2, \quad U_t(k) = \begin{pmatrix} \cos(t\log\lambda_k) & -\sin(t\log\lambda_k) \\ \sin(t\log\lambda_k) & \cos(t\log\lambda_k) \end{pmatrix}, \ \lambda_k > 1,$$
- and $(\widetilde{\mathcal{H}}_{\mathbb{R}}, \widetilde{U}_t)$ corresponds to the weakly mixing component of the orthogonal representation, thus $\widetilde{\mathcal{H}}_{\mathbb{R}}$ is either 0 or infinite dimensional.

Factoriality

The factoriality and classification problem of $M_{q,U}$ was open since 2003 for finite dimentional real orthogonal representation.

Theorem (with K. Mukherjee)

Let $t \mapsto U_t$ be a strongly continuous orthogonal representation of $\mathbb{R}$ on a real Hilbert space $\mathcal{H}_{\mathbb{R}}$ ($\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$). Then we have;

1. *If weakly mixing part of (U_t) is non-trivial, then $M_{q,U}$ is a factor and it is III_1 factor.*
2. *In-addition if $\exists\ \xi \in \mathcal{H}_{\mathbb{R}}$ s.t $U_t\xi = \xi,\ \forall t \in \mathbb{R}$ and dimension of almost periodic part is at-least 2, then it has trivial bicentralizer.*

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3. *Suppose $\exists\ \xi \in \mathcal{H}_{\mathbb{R}}$ s.t $U_t\xi = \xi,\ \forall t \in \mathbb{R}$. Then $M_{q,U}$ is a factor.*

Type Classification

Theorem (with K. Mukherjee)

Let (U_t) be real orthogonal representation on the real Hilbert space such (U_t) is almost periodic and $\dim(\mathcal{H}_{\mathbb{R}}) \geq 2$ *with an invariant real vector. Let G be the closed subgroup of $\mathbb{R}_+^{\times}$ generated by the spectrum of $A \quad (U_t = A^{it})$. Then*

$$\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)'' \text{ is } = \begin{cases} \text{type } III_1 & \text{if } G = \mathbb{R}_+ \\ \text{type } III_\lambda & \text{if } G = \lambda^{\mathbb{Z}}, \quad 0 < \lambda < 1 \\ \text{type } II_1 & \text{if } G = \{1\} \end{cases}$$

Of course, the type II_1 case corresponds to trivial (U_t) and that case we get Bozejko-Speicher II_1 factor. Notice that $S(\Gamma(\mathcal{H}_{\mathbb{R}}, U_t)'')$ is the spectrum of the modular operator since the centralizer of the free quasi-free state is a factor

Refererence:

P. Bikram and K. Mukherjee, "Generator masas in q-deformed Araki-Woods von Neumann algebras and factoriality", J. Funct. Anal. 273, 14431478.

Thanks

THANK YOU