Monojit Bhattacharjee

Characteristic Functions of Hypercontractions

Monojit Bhattacharjee

Indian Institute of Technology
Bombay

OTOA-2018
ISI,Bangalore,December 17, 2018

(A joint work with Bata Krishna Das and Jaydeb Sarkar)

Invariant Subspaces of $H^2(E)$

Let E be a separable Hilbert space of infinite dimension and $H^2(E)$ be the E-valued Hardy space, that is,

$$H^2(E) := \{f(z) = \sum_{n=0}^{\infty} a_n z^n \mid \sum_{n=0}^{\infty} \|a_n\|_E^2 < \infty, z \in \mathbb{D}\}.$$

Invariant Subspaces of $H^2(E)$

Let E be a separable Hilbert space of infinite dimension and $H^2(E)$ be the E-valued Hardy space, that is,

$$H^2(E) := \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \;\Big|\; \sum_{n=0}^{\infty} \|a_n\|_E^2 < \infty, z \in \mathbb{D} \right\}.$$

Definition 1

A closed subspace $M \subseteq H^2(E)$ is said to be invariant if $M_z M \subseteq M$, where M_z denotes the multiplication operator on $H^2(E)$.

Invariant Subspaces of $H^2(E)$

Let E be a separable Hilbert space of infinite dimension and $H^2(E)$ be the E-valued Hardy space, that is,

$$H^2(E) := \{f(z) = \sum_{n=0}^{\infty} a_n z^n \mid \sum_{n=0}^{\infty} \|a_n\|_E^2 < \infty, z \in \mathbb{D}\}.$$

Definition 1

A closed subspace $M \subseteq H^2(E)$ is said to be invariant if $M_z M \subseteq M$, where M_z denotes the multiplication operator on $H^2(E)$.

Theorem 2 (Beurling-Lax-Halmos)

A closed subspace $M \subseteq H^2(E)$ is invariant under M_z if and only if there exists a separable Hilbert space E_* and an inner multiplier Θ such that $M = \Theta H^2(E_*)$.

Sz.-Nagy-Foias model

Definition 3

A bounded linear operator T on a Hilbert space $\mathcal{H}$ is said to be in $C_{\cdot 0}$ class if $\|T^{*n}h\| \to 0$ as $n \to \infty$ for all $h \in \mathcal{H}$, that is, if $T^{*n} \to 0$ as $n \to \infty$ in the strong operator topology.

Sz.-Nagy-Foias model

Definition 3

A bounded linear operator T on a Hilbert space $\mathcal{H}$ is said to be in $C_{\cdot 0}$ class if $\|T^{*n}h\| \to 0$ as $n \to \infty$ for all $h \in \mathcal{H}$, that is, if $T^{*n} \to 0$ as $n \to \infty$ in the strong operator topology.

- The Sz.-Nagy and Foias analytic model: It says that if T is a contraction on a separable Hilbert space, and in $C_{\cdot 0}$ class, then there exists a coefficient Hilbert space $\mathcal{E}_*$ and an M_z^*-invariant closed subspace $\mathcal{Q}$ of $\mathcal{E}_*$-valued Hardy space $H^2(\mathcal{E}_*)$ such that
$$T \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}.$$

Sz.-Nagy-Foias model

Definition 3

A bounded linear operator T on a Hilbert space $\mathcal{H}$ is said to be in $C_{\cdot 0}$ class if $\|T^{*n}h\| \to 0$ as $n \to \infty$ for all $h \in \mathcal{H}$, that is, if $T^{*n} \to 0$ as $n \to \infty$ in the strong operator topology.

- The Sz.-Nagy and Foias analytic model: It says that if T is a contraction on a separable Hilbert space, and in $C_{\cdot 0}$ class, then there exists a coefficient Hilbert space $\mathcal{E}_*$ and an M_z^*-invariant closed subspace $\mathcal{Q}$ of $\mathcal{E}_*$-valued Hardy space $H^2(\mathcal{E}_*)$ such that
$$T \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}.$$

  Moreover, using Beurling-Lax-Halmos theorem, we have a Hilbert space $\mathcal{E}$ and a $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued inner multiplier $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$, such that
$$\mathcal{Q}^\perp = \Theta H^2(\mathcal{E}).$$

Characteristic Function for contraction

For a contraction $T \in \mathcal{B}(\mathcal{H})$, consider the following contractive analytic function from $\mathcal{D}_T$ to $\mathcal{D}_{T^*}$,

$$\Theta_T(z) = [-T + zD_{T^*}(1 - zT^*)^{-1}D_T]|_{\mathcal{D}_T} \quad \text{for } z \in \mathbb{D},$$

where $D_T = (I - T^*T)^{\frac{1}{2}}$, $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$, $\mathcal{D}_T = \overline{\text{ran}} D_T$ and $\mathcal{D}_{T^*} = \overline{\text{ran}} D_{T^*}$. It is known as the Characteristic Function for a contraction T. It helps to modify the Sz.-Nagy-Foias model more explicitly.

Characteristic Function for contraction

For a contraction $T \in \mathcal{B}(\mathcal{H})$, consider the following contractive analytic function from $\mathcal{D}_T$ to $\mathcal{D}_{T^*}$,

$$\Theta_T(z) = [-T + zD_{T^*}(1 - zT^*)^{-1}D_T]|_{\mathcal{D}_T} \quad \text{for } z \in \mathbb{D},$$

where $D_T = (I - T^*T)^{\frac{1}{2}}$, $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$, $\mathcal{D}_T = \overline{\text{ran}} D_T$ and $\mathcal{D}_{T^*} = \overline{\text{ran}} D_{T^*}$. It is known as the Characteristic Function for a contraction T. It helps to modify the Sz.-Nagy-Foias model more explicitly.

- Modified Sz.-Nagy-Foias model: If T is a $C_{.0}$-contraction on a seperable Hilbert space $\mathcal{H}$, then there exist a co-invariant subspace $\mathcal{Q}$ of $H^2(\mathcal{D}_{T^*})$ such that

$$T \cong P_{\mathcal{Q}} M_z|_{\mathcal{Q}}.$$

And also the co-invariant subspace $\mathcal{Q}$ can be expressed in terms of the characteristic function of T, that is,

$$\mathcal{Q}^{\perp} = \Theta_T H^2(\mathcal{D}_T).$$

Question

In this talk, we will address the following questions in multi-variable setting in the space of (weighted-) Bergman space over the Euclidean Ball in $\mathbb{C}^n$.

Question

In this talk, we will address the following questions in multi-variable setting in the space of (weighted-) Bergman space over the Euclidean Ball in $\mathbb{C}^n$. More precisely,

(1) Given a (weighted-) Bergman shift invariant subspace M, can there be an analogous representation (like in the vector-valued Hardy space) in terms of multipliers from the Drury-Arveson space?

Question

In this talk, we will address the following questions in multi-variable setting in the space of (weighted-) Bergman space over the Euclidean Ball in $\mathbb{C}^n$. More precisely,

(1) Given a (weighted-) Bergman shift invariant subspace M, can there be an analogous representation (like in the vector-valued Hardy space) in terms of multipliers from the Drury-Arveson space?

(2) Is there is any explicit description of that multiplier (analogous to the characteristic function for contraction)?

For an n-tuple of commuting bounded linear operators $T = (T_1, \dots, T_n) \in \mathcal{B}(\mathcal{H})^n$, we say T is a row contraction if the operator

$$\mathcal{H}^n \to \mathcal{H}, \; (h_1, \dots, h_n) \mapsto \sum_{i=1}^{n} T_i h_i$$

is a contraction. That is, the operator viewed as a row operator $T : \mathcal{H}^n \to \mathcal{H}$ is a contraction. Consider the associated completely positive map

$$\sigma_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), X \mapsto \sum_{i=1}^{n} T_i X T_i^*.$$

Using the map σ_T, for each $k \in \mathbb{N}$, we are going to define the defect operators of different orders of T.

m-Hypercontraction Operators

Let us consider the operator

$$\Delta_T^{(k)} = (1-\sigma_T)^k(1_{\mathcal{H}}) = \sum_{j=0}^{k}(-1)^j \left(\begin{array}{c} k \\ j \end{array} \right) \sum_{|\alpha|=j} \gamma_\alpha T^\alpha T^{*\alpha} \quad (k \in \mathbb{N}),$$

where $\gamma_\alpha = \frac{|\alpha|!}{\alpha!}$ for $\alpha \in \mathbb{N}^n$.

m-Hypercontraction Operators

Let us consider the operator

$$\Delta_T^{(k)} = (1-\sigma_T)^k(1_{\mathcal{H}}) = \sum_{j=0}^{k}(-1)^j \binom{k}{j} \sum_{|\alpha|=j} \gamma_\alpha T^\alpha T^{*\alpha} \quad (k \in \mathbb{N}),$$

where $\gamma_\alpha = \frac{|\alpha|!}{\alpha!}$ for $\alpha \in \mathbb{N}^n$.

Definition 4

An n-tuple of commuting bounded linear operators T is said to be a *m-hypercontraction* if $\Delta_T^{(1)} \geq 0$ and $\Delta_T^{(m)} \geq 0$.

For a m-hypercontraction, the defect operator of order m is $D_{m,T^*} = (\Delta_T^{(m)})^{\frac{1}{2}}$ and the defect space is $\mathcal{D}_{m,T^*} = \overline{\operatorname{ran}}(\Delta_T^{(m)})^{\frac{1}{2}}$.

m-Hypercontraction Operators

Let us consider the operator

$$\Delta_T^{(k)} = (1-\sigma_T)^k(1_{\mathcal{H}}) = \sum_{j=0}^{k}(-1)^j \begin{pmatrix} k \\ j \end{pmatrix} \sum_{|\alpha|=j} \gamma_\alpha T^\alpha T^{*\alpha} \quad (k \in \mathbb{N}),$$

where $\gamma_\alpha = \frac{|\alpha|!}{\alpha!}$ for $\alpha \in \mathbb{N}^n$.

Definition 4

An n-tuple of commuting bounded linear operators T is said to be a *m-hypercontraction* if $\Delta_T^{(1)} \geq 0$ and $\Delta_T^{(m)} \geq 0$.

For a m-hypercontraction, the defect operator of order m is $D_{m,T^*} = (\Delta_T^{(m)})^{\frac{1}{2}}$ and the defect space is $\mathcal{D}_{m,T^*} = \overline{\text{ran}}(\Delta_T^{(m)})^{\frac{1}{2}}$.

- This is also well-known that for a m-hypercontraction T, $\Delta_T^{(k)} \geq 0$ for all $1 \leq k \leq m$.

m-Hypercontraction Operators

Let us consider the operator

$$\Delta_T^{(k)} = (1-\sigma_T)^k(1_{\mathcal{H}}) = \sum_{j=0}^{k}(-1)^j \left(\begin{array}{c} k \\ j \end{array} \right) \sum_{|\alpha|=j} \gamma_\alpha T^\alpha T^{*\alpha} \quad (k \in \mathbb{N}),$$

where $\gamma_\alpha = \frac{|\alpha|!}{\alpha!}$ for $\alpha \in \mathbb{N}^n$.

Definition 4

An n-tuple of commuting bounded linear operators T is said to be a *m-hypercontraction* if $\Delta_T^{(1)} \geq 0$ and $\Delta_T^{(m)} \geq 0$.

For a m-hypercontraction, the defect operator of order m is $D_{m,T^*} = (\Delta_T^{(m)})^{\frac{1}{2}}$ and the defect space is $\mathcal{D}_{m,T^*} = \overline{\text{ran}}(\Delta_T^{(m)})^{\frac{1}{2}}$.

- This is also well-known that for a m-hypercontraction T, $\Delta_T^{(k)} \geq 0$ for all $1 \leq k \leq m$.
- An m-hypercontraction $T \in \mathcal{B}(\mathcal{H})^n$ is said to be *pure* if

$$\text{SOT} - \lim_{k\to\infty} \sigma_T^k(I_{\mathcal{H}}) = 0.$$

Weighted Bergman Space

For any positive integer $\ell \geq 0$ and a complex Hilbert space $\mathcal{E}$, we denote by $\mathbb{H}_\ell(\mathbb{B}^n, \mathcal{E})$ the $\mathcal{E}$-valued weighted Bergman space with domain $\mathbb{B}^n$, that is

$$\mathbb{H}_\ell(\mathbb{B}^n, \mathcal{E}) := \Big\{ f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha z^\alpha \in \mathcal{O}(\mathbb{B}^n, \mathcal{E}) : \|f\|^2 = \sum_{\alpha \in \mathbb{N}^n} \frac{\|f_\alpha\|^2}{\rho_\ell(\alpha)} < \infty \Big\},$$

where $\rho_\ell(\alpha) = \frac{(\ell + |\alpha| - 1)!}{\alpha!(\ell - 1)!}$.

Weighted Bergman Space

For any positive integer $\ell \geq 0$ and a complex Hilbert space $\mathcal{E}$, we denote by $\mathbb{H}_\ell(\mathbb{B}^n, \mathcal{E})$ the $\mathcal{E}$-valued weighted Bergman space with domain $\mathbb{B}^n$, that is

$$\mathbb{H}_\ell(\mathbb{B}^n, \mathcal{E}) := \Big\{ f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha z^\alpha \in \mathcal{O}(\mathbb{B}^n, \mathcal{E}) : \|f\|^2 = \sum_{\alpha \in \mathbb{N}^n} \frac{\|f_\alpha\|^2}{\rho_\ell(\alpha)} < \infty \Big\},$$

where $\rho_\ell(\alpha) = \frac{(\ell+|\alpha|-1)!}{\alpha!(\ell-1)!}$.

- It is also a reproducing kernel Hilbert space with kernel

$$K_\ell : \mathbb{B}^n \times \mathbb{B}^n \to \mathcal{B}(\mathcal{E}), \quad K_\ell(z, w) = \frac{1_{\mathcal{E}}}{(1 - \langle z, w \rangle)^\ell}.$$

- In particular, if $\ell = 1$, $\mathbb{H}_1(\mathbb{B}^n, \mathcal{E})$ is known as the Drury-Arveson space and we use $H_n^2(\mathcal{E})$ to denote it.

Model for m-Hypercontraction operators

Muller and Vasilescu have shown that a pure m-hypercontraction can be dilated to the weighted-shift operators on $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$. More precisely,

Theorem 5 (Muller and Vasilescu)

Let T be a pure m-hypercontraction on a Hilbert space $\mathcal{H}$. Then there exists a co-invariant subspace $\mathcal{Q}$ of $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ such that $T \cong (P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}} M_{z_n}|_{\mathcal{Q}})$.

Model for m-Hypercontraction operators

Muller and Vasilescu have shown that a pure m-hypercontraction can be dilated to the weighted-shift operators on $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$. More precisely,

Theorem 5 (Muller and Vasilescu)

Let T be a pure m-hypercontraction on a Hilbert space $\mathcal{H}$. Then there exists a co-invariant subspace $\mathcal{Q}$ of $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ such that $T \cong (P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}} M_{z_n}|_{\mathcal{Q}})$.

They established this result with the help of the following dilation map, $\pi_m : \mathcal{H} \to \mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ defined by

$$\pi_m h(z) = \sum_{\alpha \in \mathbb{N}^n} \rho_m(\alpha)(D_{m,T^*} T^{*\alpha} h) z^\alpha = D_{m,T^*}(1 - ZT^*)^{-m} h,$$

where $Z : \mathcal{H}^n \to \mathcal{H}$ defined by $Z(h_1, \ldots, h_n) = \sum_{i=1}^n z_i h_i$ and M_{z_i} is the shift on $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ for all $i = 1, \ldots, n$ and $\mathcal{Q} = \text{ran}\pi_m$ which is a joint co-invariant subspace of $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$.

Invariant Subspaces of Weighted Bergman space

Recently, Sarkar has produced a Beurling-Lax-Halmos type result for the general reproducing kernel Hilbert space. We will invoke this result for the weighted Bergman space.

Theorem 6 (Sarkar)

Let M be a closed subspace of $\mathbb{H}_m(\mathbb{B}^n, \mathcal{E})$. Then M is a joint $(M_{z_1} \otimes I_{\mathcal{E}}, \ldots, M_{z_n} \otimes I_{\mathcal{E}})$-invariant subspace if and only if there exists a Hilbert space $\mathcal{E}_*$ and a partial isometric multiplier Φ from $H^2_n(\mathcal{E}_*)$ to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{E})$ such that $M = \Phi H^2_n(\mathcal{E}_*)$.

Recently, Sarkar has produced a Beurling-Lax-Halmos type result for the general reproducing kernel Hilbert space. We will invoke this result for the weighted Bergman space.

Theorem 6 (Sarkar)

Let M be a closed subspace of $\mathbb{H}_m(\mathbb{B}^n, \mathcal{E})$. Then M is a joint $(M_{z_1} \otimes I_{\mathcal{E}}, \ldots, M_{z_n} \otimes I_{\mathcal{E}})$-invariant subspace if and only if there exists a Hilbert space $\mathcal{E}_*$ and a partial isometric multiplier Φ from $H_n^2(\mathcal{E}_*)$ to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{E})$ such that $M = \Phi H_n^2(\mathcal{E}_*)$.

- Combining above two results we have that given a m-hypercontraction T there exists a Hilbert space $\mathcal{E}_*$ and a partial isometric multiplier Φ from $H_n^2(\mathcal{E}_*)$ to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{E})$ such that $T_i \cong P_{(\Phi H_n^2(\mathcal{E}_*))^\perp} M_{z_i}|_{(\Phi H_n^2(\mathcal{E}_*))^\perp}$, for $i = 1, \ldots, n$.

Construction of the Characteristic function

- We consider the operator $C_{m,T} : \mathcal{H} \to l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})$ defined by
$$h \mapsto (\rho_{m-1}(\alpha)^{\frac{1}{2}} D_{m,T^*} T^{*\alpha} h)_{\alpha \in \mathbb{N}^n}.$$

Construction of the Characteristic function

- We consider the operator $C_{m,T} : \mathcal{H} \to l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})$ defined by

$$h \mapsto (\rho_{m-1}(\alpha)^{\frac{1}{2}} D_{m,T^*} T^{*\alpha} h)_{\alpha \in \mathbb{N}^n}.$$

- Here, $\begin{bmatrix} T^* \\ C_{m,T} \end{bmatrix} : \mathcal{H} \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})$ is an isometry.

Construction of the Characteristic function

- We consider the operator $C_{m,T} : \mathcal{H} \to l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})$ defined by
$$h \mapsto (\rho_{m-1}(\alpha)^{\frac{1}{2}} D_{m,T^*} T^{*\alpha} h)_{\alpha \in \mathbb{N}^n}.$$
- Here, $\begin{bmatrix} T^* \\ C_{m,T} \end{bmatrix} : \mathcal{H} \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})$ is an isometry.

Definition 7 (Characteristic triple)

For a m-hypercontraction T on $\mathcal{H}$, a triple $(\mathcal{E}, B, D)$ consists of a Hilbert space $\mathcal{E}$ and operators $B \in \mathcal{B}(\mathcal{E}, \mathcal{H}^n)$, $D \in \mathcal{B}\big(\mathcal{E}, l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})\big)$ is a characteristic triple if the corresponding block operator matrix

$$\begin{bmatrix} T^* & B \\ C_{m,T} & D \end{bmatrix} : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*})$$

is a co-isometry.

Construction of the Characteristic function

For a characteristic triple $(\mathcal{E}, B, D)$ of T, let us define an operator-valued analytic function $\Phi : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}, \mathcal{D}_{m,T^*})$ by

$$\Phi(z) = \sum_{\alpha \in \mathbb{N}^n} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_\alpha z^\alpha + D_{m,T^*}(1 - ZT^*)^{-m} ZB \qquad \big(z \in \mathbb{B}^n\big).$$

Construction of the Characteristic function

For a characteristic triple $(\mathcal{E}, B, D)$ of T, let us define an operator-valued analytic function $\Phi : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}, \mathcal{D}_{m,T^*})$ by

$$\Phi(z) = \sum_{\alpha \in \mathbb{N}^n} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_\alpha z^\alpha + D_{m,T^*}(1 - ZT^*)^{-m} ZB \qquad \big(z \in \mathbb{B}^n\big).$$

It satisfies the following identity

$$\frac{I_{\mathcal{D}_{m,T^*}}}{(1 - \langle z, w\rangle)^m} - \frac{\Phi(z)\Phi(w)^*}{1 - \langle z, w\rangle} = D_{m,T^*}(1 - ZT^*)^{-m}(1 - TW^*)^{-m} D_{m,T^*}.$$

Construction of the Characteristic function

For a characteristic triple $(\mathcal{E}, B, D)$ of T, let us define an operator-valued analytic function $\Phi : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}, \mathcal{D}_{m,T^*})$ by

$$\Phi(z) = \sum_{\alpha \in \mathbb{N}^n} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_\alpha z^\alpha + D_{m,T^*}(1 - ZT^*)^{-m} ZB \qquad \big(z \in \mathbb{B}^n\big).$$

It satisfies the following identity

$$\frac{I_{\mathcal{D}_{m,T^*}}}{(1 - \langle z, w \rangle)^m} - \frac{\Phi(z)\Phi(w)^*}{1 - \langle z, w \rangle} = D_{m,T^*}(1 - ZT^*)^{-m}(1 - TW^*)^{-m} D_{m,T^*}.$$

- Thus we have find a general recipe of constructing characteristic functions of m-hypercontractions as follows.

Theorem 8

Let T be a pure m-hypercontraction on $\mathcal{H}$. Suppose $(\mathcal{E}, B, D)$ is a characteristic triple of T. Then the above defined function Φ defines a partial isometric multiplier from $H^2_n(\mathcal{E})$ to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ such that $\mathcal{Q}^\perp = \Phi H^2_n(\mathcal{E})$, where $\mathcal{Q}$ is the model space for T.

Unitary Equivalence

Two row contractions $T = (T_1, \cdots, T_n)$ and $R = (R_1, \ldots, R_n)$ on $\mathcal{H}$ are said to be unitary equivalent if there exist a unitary U on $\mathcal{H}$ such that $T_i = UR_iU^*$, for all $i = 1, \ldots, n$.

Unitary Equivalence

Two row contractions $T = (T_1, \cdots, T_n)$ and $R = (R_1, \ldots, R_n)$ on $\mathcal{H}$ are said to be unitary equivalent if there exist a unitary U on $\mathcal{H}$ such that $T_i = UR_iU^*$, for all $i = 1, \ldots, n$.

Definition 9

The characteristic functions Φ_T and Φ_R of two pure m-hypercontractions T and R are said to coincide if there exists two unitary $\Gamma : (\mathrm{Ker} M_{\Phi_R})^\perp \to (\mathrm{Ker} M_{\Phi_T})^\perp$ and $\tau : \mathcal{D}_{m,T^*} \to \mathcal{D}_{m,R^*}$ such that

$$M_{\Phi_R}|_{(\mathrm{Ker} M_{\Phi_R})^\perp} = (I \otimes \tau) M_{\Phi_T} \Gamma.$$

Unitary Equivalence

Two row contractions $T = (T_1, \cdots, T_n)$ and $R = (R_1, \ldots, R_n)$ on $\mathcal{H}$ are said to be unitary equivalent if there exist a unitary U on $\mathcal{H}$ such that $T_i = UR_iU^*$, for all $i = 1, \ldots, n$.

Definition 9

The characteristic functions Φ_T and Φ_R of two pure m-hypercontractions T and R are said to coincide if there exists two unitary $\Gamma : (\mathrm{Ker} M_{\Phi_R})^{\perp} \to (\mathrm{Ker} M_{\Phi_T})^{\perp}$ and $\tau : \mathcal{D}_{m,T^*} \to \mathcal{D}_{m,R^*}$ such that

$$M_{\Phi_R}|_{(\mathrm{Ker} M_{\Phi_R})^{\perp}} = (I \otimes \tau) M_{\Phi_T} \Gamma.$$

Theorem 10

Two pure m-hypercontractions are unitarily equivalent if and only if their characteristic functions coincide.

Factorization w.r.t invariant subspace

- Sz.-Nagy-Foias have produced a factorization of the characteristic function for a contraction in terms of multipliers of invariant subspaces of it. A similar kind of factorization result is also true for m-hypercontractions.

Factorization w.r.t invariant subspace

- Sz.-Nagy-Foias have produced a factorization of the characteristic function for a contraction in terms of multipliers of invariant subspaces of it. A similar kind of factorization result is also true for m-hypercontractions.

Theorem 11

Let $T = (T_1, \dots, T_n)$ be a pure m-hypercontraction on $\mathcal{H}$ and $\mathcal{H}_1 \subseteq \mathcal{H}$ be a joint T-invariant subspace. Then there exists two Hilbert spaces $\mathcal{E}_T$ and $\mathcal{E}$, and two multipliers Φ_1 from $H^2_n(\mathcal{E}_T)$ to $H^2_n(\mathcal{E}_2)$ and Φ_2 from $H^2_n(\mathcal{E}_2)$ to $\mathbb{H}_m(\mathbb{B}^n, \mathcal{D}_{m,T^*})$ such that the characteristic function of T has a factorization, that is,

$$\Phi_T = \Phi_2 \Phi_1.$$

Consider $\Psi_\beta(z) = \left[\cdots, \frac{1}{\sqrt{\gamma_{|\alpha|}}}\left(\frac{|\alpha|!}{\alpha!}\right)^{\frac{1}{2}} z^\alpha I_{\mathcal{D}_{m,T^*}}, \cdots\right]_{\alpha\in\mathbb{N}^n}$, where $\gamma_j^{-1} = \begin{pmatrix} m+j-2 \\ j \end{pmatrix}$ for $j \neq 0$ and $\gamma_0 = 0$.

Theorem 12

Let T be a m-hypercontraction, and let

$$S(z) = \sum_{\alpha\in\mathbb{N}^n} \rho_{m-1}(\alpha)^{\frac{1}{2}} D_\alpha z^\alpha + D_{m,T^*}(1 - ZT^*)^{-m} ZB$$

be a characteristic function of T corresponding to a triple $(\mathcal{E}, B, D)$. Then $S(z) = \Psi_\beta(z)\tilde{S}(z)$, where $\tilde{S}(z) = D + C_{m,T}(1 - ZT^*)^{-1}ZB$ is the transfer function of the canonical co-isometry (w.r.t. the triple $(\mathcal{E}, B, D)$)

$$\begin{bmatrix} T^* & B \\ C_{m,T} & D \end{bmatrix} : \mathcal{H} \oplus \mathcal{E} \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{m,T^*}).$$

Relation between different order hypercontraction

Let T be a pure m_2-hypercontraction. Then for $1 \leq m_1 \leq m_2$, T is also a m_1-hypercontraction. For $i = 1, 2$, $\tilde{S}_i$ be the the transfer function of the co-isometries

$$\begin{bmatrix} T^* & B_i \\ C_{m_i,T} & D_i \end{bmatrix} : \mathcal{H} \oplus \mathcal{E}_i \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{m_i,T^*}).$$

Let T be a pure m_2-hypercontraction. Then for $1 \leq m_1 \leq m_2$, T is also a m_1-hypercontraction. For $i = 1, 2$, $\tilde{S}_i$ be the the transfer function of the co-isometries

$$\begin{bmatrix} T^* & B_i \\ C_{m_i,T} & D_i \end{bmatrix} : \mathcal{H} \oplus \mathcal{E}_i \to \mathcal{H}^n \oplus l^2(\mathbb{N}^n, \mathcal{D}_{m_i,T^*}).$$

Theorem 13

Let T be a pure m_2-hypercontraction. Suppose that $(\mathcal{E}_1, B_1, D_1)$ and $(\mathcal{E}_2, B_2, D_2)$ are characteristic triples of T considered as a m_1-hypercontraction and m_2-hypercontraction $(1 \leq m_1 < m_2)$. Then there exist isometries $Y \in \mathcal{B}(\overline{ran}\ C_{m_2,T}; l^2(\mathbb{N}^n, \mathcal{D}_{m_1,T^*}))$ and $X \in \mathcal{B}(\overline{ran}\ B_1^*; \mathcal{E}_2)$ such that

$$\tilde{S}_1(z)|_{(KerB_1)^\perp} = Y\tilde{S}_2(z)X$$

for all $z \in \mathbb{B}^n$.

References

[1] J. Agler, *Hypercontractions and subnormality*, J. Operator Theory 13 (1985), 203–217.

[2] V. Muller and F.-H. Vasilescu, *Standard models for some commuting multioperators*, Proc. Amer. Math. Soc. 117 (1993), 979–989.

[3] J. Sarkar, *An invariant subspace theorem and invariant subspaces of analytic reproducing kernel Hilbert spaces - I*, J. Operator Theory 73 (2015), 433–441.

[4] J. Sarkar, *An invariant subspace theorem and invariant subspaces of analytic reproducing kernel Hilbert spaces - II*, Complex Analysis and Operator Theory 10(2016), 769–782.

THANK YOU FOR YOUR ATTENTION