Thompson isometries and applications

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Banach-Stone (1932, 1937)

Let X, Y be compact Hausdorff spaces and $\phi : C(X) \to C(Y)$ be a surjective linear isometry. Then there exists a homeomorphism $\varphi : Y \to X$ and a continuous scalar function τ on Y with values of modulus 1 such that

$$\phi(f) = \tau \cdot f \circ \varphi, \quad f \in C(X).$$

Observe that the transformations $f \mapsto f \circ \varphi$ are exactly the algebra isomorphisms between $C(X)$ and $C(Y)$.

Noncommutative generalization:

Kadison (1951)

Let $\mathcal{A}, \mathcal{B}$ be (unital) C^*-algebras. If $\phi : \mathcal{A} \to \mathcal{B}$ is a surjective linear isometry, then it is of the form

$$\phi(A) = UJ(A), \quad A \in \mathcal{A},$$

where $U \in \mathcal{B}$ is a unitary and $J : \mathcal{A} \to \mathcal{B}$ is a Jordan *-isomorphism.

Let $\mathcal{A}, \mathcal{B}$ be complex algebras. The linear map $J : \mathcal{A} \to \mathcal{B}$ is called a Jordan homomorphism if it satisfies

$$J(A^2) = J(A)^2, \quad A \in \mathcal{A},$$

or, equivalently,

$$J(AB + BA) = J(A)J(B) + J(B)J(A), \quad A, B \in \mathcal{A}.$$

If $\mathcal{A}, \mathcal{B}$ are *-algebras, the Jordan homomorphism $J : \mathcal{A} \to \mathcal{B}$ is called Jordan *-homomorphism if it also satisfies $J(A^*) = J(A)^*$ for all $A \in \mathcal{A}$.

A bijective Jordan (*-)homomorphism is called Jordan (*-)isomorphism.

Connection to (*-)isomorphisms and (*-)antiisomorphisms:

Herstein's theorem: A Jordan homomorphism onto a prime algebra is either a homomorphism or an antihomomorphism.

There are several results in the literature stating roughly that in certain cases Jordan homomorphisms are direct sums of homomorphisms and antihomomorphisms.

Thompson metric (or Thompson part metric): Let $\mathcal{A}$ be a C^*-algebra $\mathcal{A}$ and $\mathcal{A}^{++}$ its positive definite cone. The metric d_T on $\mathcal{A}^{++}$ is defined as follows:

$$d_T(A, B) = \log \max\{M(A/B), M(B/A)\}, \quad A, B \in \mathcal{A}^{++},$$

where $M(X/Y) = \inf\{t > 0 : X \leq tY\}$ for any $X, Y \in \mathcal{A}^{++}$. It is easy to see that d_T can also be rewritten as

$$d_T(A, B) = \left\| \log \left(A^{-1/2} B A^{-1/2} \right) \right\|, \quad A, B \in \mathcal{A}^{++}.$$

Shortest path distance in an appropriate Finsler-type geometry on $\mathcal{A}^{++}$, a number of applications in several areas.

The structure of Thompson isometries:

Theorem 1 (O. Hatori and M (2014))

*The surjective map $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ is a Thompson isometry iff there is a central projection P in $\mathcal{B}$ and a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$ such that ϕ is of the form*

$$\phi(A) = \phi(1)^{1/2} \left(PJ(A) + (1-P)J(A^{-1})\right) \phi(1)^{1/2}, \quad A \in \mathcal{A}^{++}. \tag{1}$$

Sketch of the proof:
First of all, one can show that for any given element $Y \in \mathcal{B}^{++}$, the map $B \mapsto YBY$ is a Thompson isometry of the positive definite cone $\mathcal{B}^{++}$. Therefore, multiplying ϕ by $\phi(1)^{-1/2}$ from both sides, we can and do assume that ϕ is unital, i.e., $\phi(1) = 1$ holds.

The classical Mazur-Ulam theorem says that surjective isometries between real normed linear spaces are affine, they preserve the operation of the arithmetic mean. The argument to show that result can essentially be applied to show the interesting fact that Thompson isometries necessarily preserve the operation of the geometric mean which is defined by

$$A \sharp B = A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^{1/2} A^{1/2} \tag{2}$$

for any elements A, B in a positive definite cone.

It is a classical result (sometimes referred to as Anderson-Trapp theorem) that $A \sharp B$ is the unique positive invertible solution of the simple equation $XA^{-1}X = B$. It follows easily that ϕ preserves the operation appearing on the left hand side of this equation (which turns to be a sort of abstract reflection), i.e., we have $\phi(XA^{-1}X) = \phi(X)\phi(A)^{-1}\phi(X)$, $A, X \in \mathcal{A}^{++}$. Since ϕ sends the unit to the unit, we obtain that ϕ preserves the Jordan triple product $(A, B) \mapsto ABA$ of positive invertible elements, i.e., satisfies

$$\phi(ABA) = \phi(A)\phi(B)\phi(A), \qquad A, B \in \mathcal{A}^{++}.$$

This implies that $\phi(A^r) = \phi(A)^r$ holds for all $A \in \mathcal{A}^{++}$ and rational numbers r.

One can easily see that the topology what the Thompson metric generates coincides with the topology what the C^*-norm generates on the positive definite cone. Therefore, ϕ is continuous with respect to the norm topology. It then follows that $\phi(A^r) = \phi(A)^r$ holds for all $A \in \mathcal{A}^{++}$ and real number r.

The so-called symmetrized Lie-Trotter formula asserts that

$$\left(e^{-(t/2)X} e^{tY} e^{-(t/2)X}\right)^{1/t} \xrightarrow{t \to 0} e^{Y-X}$$

in norm which implies that

$$\frac{d_T\left(e^{tX}, e^{tY}\right)}{t} \xrightarrow{t \to 0} \|X - Y\| \tag{3}$$

holds for any $X, Y \in \mathcal{A}_s$.

Define $F : \mathcal{A}_s \to \mathcal{B}_s$ by $F(X) = \log \phi(e^X)$, $X \in \mathcal{A}_s$. Then we have

$$e^{tF(X)} = (e^{F(X)})^t = \phi(e^X)^t = \phi((e^X)^t) = \phi(e^{tX}), \quad t > 0, X \in \mathcal{A}_s.$$

Applying the formula (3) and using the fact that ϕ is a Thompson isometry, we deduce that

$$\|F(X) - F(Y)\| = \|X - Y\|, \quad X, Y \in \mathcal{A}_s.$$

Therefore, $F : \mathcal{A}_s \to \mathcal{B}_s$ is a surjective isometry which also satisfies $F(0) = 0$. By the classical Mazur-Ulam theorem it follows that F is a linear surjective isometry between the self-adjoint parts of C^*-algebras.

A result of Kadison describes the structure of those transformations. Namely, it tells that there is a central self-adjoint unitary element S in $\mathcal{A}$ and a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{A}$ such that

$$F(X) = SJ(X), \quad X \in \mathcal{A}_s.$$

Clearly, $S = 2P - 1$ holds with some central projection $P \in \mathcal{B}$.

The rest of the proof of the necessity part of the theorem is simple calculation and so is its sufficiency part.

The following corollary is especially important for us.

Corollary 2

Let $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ be an order isomorphism, i.e., a surjective mapping with the property that for any $A, B \in \mathcal{A}^{++}$ we have

$$A \leq B \Longleftrightarrow \phi(A) \leq \phi(B).$$

(Observe that this property obviously implies the injectivity of ϕ.) If ϕ is also positive homogeneous (i.e., satisfies $\phi(\lambda A) = \lambda \phi(A)$ for all $A \in \mathcal{A}^{++}$ and positive number λ), then ϕ is necessarily of the form

$$\phi(A) = \phi(1)^{1/2} J(A) \phi(1)^{1/2}, \quad A \in \mathcal{A}^{++},$$

*where $J : \mathcal{A} \to \mathcal{B}$ is a Jordan *-isomorphism.*

Hence, homogeneous order isomorphisms are automatically additive.

Remark: Employing a different approach using, among several others,

- an abstract Mazur-Ulam type result for noncommutative structures (e.g., for so-called point reflection geometries) and
- structural results on commutativity preserving linear bijections between self-adjoint parts of prime *-algebras

we described the surjective maps between positive definite cones in factor von Neumann algebras which preserve

$$(A, B) \mapsto N(f(A^{-1/2}BA^{-1/2}))$$

type generalized distance measures. Here N is a complete symmetric norm and $f :]0, \infty[\to \mathbb{R}$ is a continuous function which takes the value 0 exactly at 1 and also satisfies $|f(y^2)| \geq K|f(y)|$ for some $K > 1$ in a neighborhood of 1.

Examples for such distance measures include: Stein's loss, symmetric Stein divergence, Jeffrey's Kullback–Leibler divergence, log-determinant α-divergence.

Relative entropies play a very important role in classical information theory, they are used to measure how well a probability distribution approximates another one. For any two given probability distributions p, q on a finite set $\mathcal{X}$, their Rényi α-divergence ($\alpha \in]0, \infty[, \alpha \neq 1$) is the quantity

$$D_\alpha(p||q) = \frac{1}{\alpha - 1} \log \sum_{x \in \mathcal{X}} p(x)^\alpha q(x)^{1-\alpha}.$$

Among all relative entropies, the parametric family $D_\alpha(.||.)$ has a distinguished role for several reasons. Its elements have various desirable mathematical properties: they are non-increasing under stochastic maps, jointly convex for $\alpha \in]0, 1[$, jointly quasi-convex for $\alpha \in]1, \infty[$, monotone increasing as a function of the parameter α, and the Kullback-Leibler divergence (i.e., the most classical relative entropy) is their limiting case as $\alpha \to 1$, etc.

In what follows we consider several variants of quantum Rényi divergences and their symmetries. Motivation comes from Wigner's theorem on the description of quantum mechanical symmetry transformations (maps on pure states preserving transition probability).

The next results are from our recent paper

L. Molnár, *Quantum Rényi relative entropies: their symmetries and their essential difference,* submitted.

If $\mathcal{A}$ is a C^*-algebra and τ is a positive linear functional on $\mathcal{A}$ such that $\tau(XY) = \tau(YX)$ holds for all $X, Y \in \mathcal{A}$, then τ is called a trace. We say that the trace τ on $\mathcal{A}$ is faithful if $\tau(A) = 0$, $A \in \mathcal{A}^+$ implies $A = 0$.

Density space in a C^*-algebra $\mathcal{A}$ with faithful trace τ (Farenick etal, 2016/17):

$$\mathcal{D}_\tau(\mathcal{A}) = \{A \in \mathcal{A}^+ \ : \ \tau(A) = 1\}.$$

To avoid technical difficulties, we consider only nonsingular densities. For any C^*-algebra $\mathcal{A}$ with faithful trace τ, we set

$$\mathcal{D}_\tau^{-1}(\mathcal{A}) = \mathcal{D}_\tau(\mathcal{A}) \cap \mathcal{A}^{++}.$$

There are many ways to extend the classical quantity

$$D_\alpha(p||q) = \frac{1}{\alpha - 1} \log \sum_{x \in \mathcal{X}} p(x)^\alpha q(x)^{1-\alpha}$$

to the quantum setting. A list of those five which are very strongly studied in quantum information theory is the following (here we define them on density spaces of C^*-algebras closely following the definitions given in the literature for density matrices). Let $\alpha \in]0, \infty[, \alpha \neq 1$.

The conventional Rényi relative entropy is defined by Petz as

$$\tau\text{-}D_\alpha^c(A||B) = \frac{1}{\alpha - 1} \log \tau(A^\alpha B^{1-\alpha}), \quad A, B \in \mathcal{D}_\tau^{-1}(\mathcal{A}). \tag{4}$$

The minimal (or sandwiched) Rényi relative entropy is the quantity

$$\tau\text{-}D_\alpha^{min}(A||B) = \frac{1}{\alpha - 1} \log \tau \left(B^{\frac{1-\alpha}{2\alpha}} A B^{\frac{1-\alpha}{2\alpha}} \right)^\alpha, \quad A, B \in \mathcal{D}_\tau^{-1}(\mathcal{A}) \tag{5}$$

due to Müller-Lennert, Dupuis, Szehr, Fehr and Tomamichel.

These two sorts of relative entropies are particular cases of the $\alpha - z$-Rényi relative entropies

$$\tau\text{-}D_{\alpha,z}(A||B) = \frac{1}{\alpha - 1} \log \tau \left(B^{\frac{1-\alpha}{2z}} A^{\frac{\alpha}{z}} B^{\frac{1-\alpha}{2z}} \right)^{z}, \quad a, b \in \mathcal{D}_{\tau}^{-1}(\mathcal{A}) \tag{6}$$

introduced by Audenaert and Datta. Here $z > 0$ is any positive real number and, clearly, if $z = 1$, then we get the conventional Rényi divergence, while in the case $z = \alpha$, we obtain the minimal Rényi relative entropy.

The maximal Rényi relative entropy is defined by

$$\tau\text{-}D_{\alpha}^{max}(A||B) = \frac{1}{\alpha - 1} \log \tau \left(B^{1/2} (B^{-1/2} A B^{-1/2})^{\alpha} B^{1/2} \right), A, B \in \mathcal{D}_{\tau}^{-1}(\mathcal{A}) \tag{7}$$

and was considered by Petz and Ruskai.

Mosonyi and Ogawa introduced another type of Rényi relative entropy which, in our present setting, is defined as

$$\tau\text{-}D_{\alpha}^{mo}(A||B) = \frac{1}{\alpha - 1} \log \tau \left(\exp(\alpha \log A + (1 - \alpha) \log B) \right), A, B \in \mathcal{D}_{\tau}^{-1}(\mathcal{A}). \tag{8}$$

The above definitions can of course be extended from $\mathcal{D}_\tau^{-1}(\mathcal{A})$ to the whole positive definite cone $\mathcal{A}^{++}$. We prefer to work with the following modified numerical quantities (we erase the multiplicator $\frac{1}{\alpha-1}$ and the function log):

$$\tau\text{-}Q_\alpha^c(A||B) = \tau(A^\alpha B^{1-\alpha}), \tag{9}$$

$$\tau\text{-}Q_\alpha^{min}(A||B) = \tau\left(B^{\frac{1-\alpha}{2\alpha}} A B^{\frac{1-\alpha}{2\alpha}}\right)^\alpha, \tag{10}$$

$$\tau\text{-}Q_{\alpha,z}(A||B) = \tau\left(b^{\frac{1-\alpha}{2z}} A^{\frac{\alpha}{z}} B^{\frac{1-\alpha}{2z}}\right)^z, \tag{11}$$

$$\tau\text{-}Q_\alpha^{max}(A||B) = \tau\left(B^{1/2}(B^{-1/2}AB^{-1/2})^\alpha B^{1/2}\right), \tag{12}$$

$$\tau\text{-}Q_\alpha^{mo}(A||B) = \tau\left(\exp(\alpha \log A + (1-\alpha)\log B)\right) \tag{13}$$

for all $A, B \in \mathcal{A}^{++}$.

The transformations between positive definite cones of C^*-algebras which preserve the above defined Rényi relative entropy related quantities are completely described in the following result.

Theorem 3

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with faithful traces τ, ω, respectively, and let α, z be positive numbers, $\alpha \neq 1$. Let $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ be a surjective map. Then ϕ satisfies

$$\omega\text{-}Q_{\alpha,z}(\phi(A)||\phi(B)) = \tau\text{-}Q_{\alpha,z}(A||B), \quad A, B \in \mathcal{A}^{++} \tag{14}$$

if and only if
*there is a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$ and an element $C \in \mathcal{B}^{++}$ central in the algebra $\mathcal{A}$ such that*

$$\phi(A) = CJ(A), \quad A \in \mathcal{A}^{++}$$

and $\omega(CJ(X)) = \tau(X)$ holds for all $X \in \mathcal{A}$.
Analogous assertions are valid for all other quantities in (9)-(13), for every positive number α different from 1 with the exception of the quantity in (12) where we need to assume that $\alpha \leq 2$.

As a corollary, we easily obtain the following description of the structure of maps between density spaces of C^*-algebras preserving the different types of quantum Rényi relative entropy.

Corollary 4

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with faithful traces τ, ω, respectively, and let α, z be positive real numbers, $\alpha \neq 1$. Assume that $\phi : \mathcal{D}_\tau^{-1}(\mathcal{A}) \to \mathcal{D}_{\tau'}^{-1}(\mathcal{B})$ is a surjective map. Then ϕ satisfies

$$\omega\text{-}D_{\alpha,z}(\phi(A)||\phi(B)) = \tau\text{-}D_{\alpha,z}(A||B), \quad A, B \in \mathcal{D}_\tau^{-1}(\mathcal{A})$$

*if and only if there is a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$ and an element $C \in \mathcal{B}^{++}$ central in the algebra $\mathcal{A}$ such that $\phi(A) = CJ(A)$, $A \in \mathcal{D}_\tau^{-1}(\mathcal{A})$ and $\omega(CJ(X)) = \tau(X)$ holds for all $X \in \mathcal{A}$.*
Analogous assertions are valid for all other quantities in (4)-(8), for every positive number α different from 1 with the exception of (7), where we need to assume that $\alpha \leq 2$.

About the proof of the former result (maps between positive definite cones):

The key idea is the following. We characterize the order by the different quantum Rényi relative entropies. We show that the transformations between positive definite cones respecting those relative entropies are, after some necessary modifications, positive homogeneous order isomorphisms.

We apply the corollary of our theorem on Thompson isometries where the structure of such transformations is given.

These are just the basic ideas, the details are much more complicated and use several other tools.

The characterizations of order in terms of the different types of Rényi relative entropy related quantities.

Lemma 5

Let $\alpha \in]0,\infty[$ be a real number, $\mathcal{A}$ be a C^*-algebra with a faithful trace τ, and select $A, B \in \mathcal{A}^{++}$. We have

$$A \leq B \Longleftrightarrow \tau\left((XAX)^{\alpha}\right) \leq \tau\left((XBX)^{\alpha}\right), \quad X \in \mathcal{A}^{++}$$

.

Therefore, for any given real number $z > 0$, we have that

$$A^{\frac{\alpha}{z}} \leq B^{\frac{\alpha}{z}} \Longleftrightarrow \tau\text{-}Q_{\alpha,z}(A||X) \leq \tau\text{-}Q_{\alpha,z}(B||X), \quad X \in \mathcal{A}^{++},$$

and we have

$$A^{\frac{1-\alpha}{z}} \leq B^{\frac{1-\alpha}{z}} \Longleftrightarrow \tau\text{-}Q_{\alpha,z}(X||A) \leq \tau\text{-}Q_{\alpha,z}(X||B), \quad X \in \mathcal{A}^{++}.$$

For the characterization of the order by the quantity τ-$Q_\alpha^{max}(.||.)$ we need the following observation.

Lemma 6

Let $f :]0,\infty[\to \mathbb{R}$ be a strictly increasing continuous function which is also operator monotone. Assume that the function $g : [0,\infty[\to \mathbb{R}$ defined by $g(t) = tf(t)$ for $t > 0$ and $g(0) = 0$ is continuous on $[0,\infty[$. Let $\mathcal{A}$ be a C^*-algebra with faithful trace τ and select $A, B \in \mathcal{A}^{++}$. Then

$$A^2 \leq B^2 \iff \tau(Xf(XA^2X)X) \leq \tau(Xf(XB^2X)X), \quad X \in \mathcal{A}^{++}.$$

Using the previous lemma, we have the following

Lemma 7

Let $\alpha \in]0, 2]$, $\alpha \neq 1$ *be a real number,* $\mathcal{A}$ *be a* C^**-algebra with a faithful trace* τ, *and select* $A, B \in \mathcal{A}^{++}$. *For* $0 < \alpha < 1$ *we have*

$$A \leq B \iff \tau\text{-}Q_\alpha^{max}(A||X) \leq \tau\text{-}Q_\alpha^{max}(B||X), \quad X \in \mathcal{A}^{++},$$

while for $1 < \alpha \leq 2$ *we have*

$$A \leq B \iff \tau\text{-}Q_\alpha^{max}(X||B) \leq \tau\text{-}Q_\alpha^{max}(X||A), \quad X \in \mathcal{A}^{++}.$$

Next, characterization of order in terms of the quantity $\tau\text{-}Q_\alpha^{mo}(.||.)$. In what follows, $\mathcal{A}_s$ denotes the space of all self-adjoint elements of the C^*-algebra $\mathcal{A}$.

Lemma 8

Let $\mathcal{A}$ be a C^*-algebra with a faithful trace τ. Pick $T, S \in \mathcal{A}_s$. We have

$$T \leq S \iff \tau(\exp(T+X)) \leq \tau(\exp(S+X)), \quad X \in \mathcal{A}_s.$$

In particular, for any $\alpha \in]0,\infty[$, $\alpha \neq 1$ and for arbitrary $A, B \in \mathcal{A}^{++}$, we have

$$\log A \leq \log B \iff \tau\text{-}Q_\alpha^{mo}(A||X) \leq \tau\text{-}Q_\alpha^{mo}(B||X), \quad X \in \mathcal{A}^{++}.$$

Moreover, if $\alpha > 1$, then for any $A, B \in \mathcal{A}^{++}$ we have

$$\log A \leq \log B \iff \tau\text{-}Q_\alpha^{mo}(X||B) \leq \tau\text{-}Q_\alpha^{mo}(X||A), \quad X \in \mathcal{A}^{++}.$$

Beside characterizations of the order, we will also need some sufficient conditions for centrality of positive definite elements.

Lemma 9

*Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with faithful traces τ and ω, respectively. Let $J : \mathcal{A} \to \mathcal{B}$ be a Jordan *-isomorphism, $C \in \mathcal{B}^{++}$ and β be a positive real number such that $\omega\left((CJ(A^{\frac{1}{\beta}})C)^\beta\right) = \tau(A)$ holds for all $A \in \mathcal{A}^{++}$. Then C is necessarily a central element in $\mathcal{B}$.*

Another condition for centrality that we need:

Lemma 10

*Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with faithful traces τ and ω, respectively. Let $J : \mathcal{A} \to \mathcal{B}$ be a Jordan *-isomorphism and $X_0 \in \mathcal{B}_s$ be a self-adjoint element such that*

$$\omega(\exp(J(T) + X_0)) = \tau(\exp(T)) \tag{15}$$

holds for all $T \in \mathcal{A}_s$. Then X_0 is necessarily a central element in $\mathcal{B}$.

Sketch of the proof of Theorem 3:
We deal only with the necessity parts (sufficiency parts are easy to check using properties of Jordan *-isomorphisms).

I. We assume that $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ is a surjective map which satisfies

$$\omega\text{-}Q_{\alpha,z}(\phi(A)||\phi(B)) = \tau\text{-}Q_{\alpha,z}(A||B), \quad A, B \in \mathcal{A}^{++}.$$

By the corresponding characterization of order (Lemma 5), we have that ϕ has the following order preserving property: for any pair of elements $A, B \in \mathcal{A}^{++}$, we have $A^{\frac{\alpha}{z}} \leq B^{\frac{\alpha}{z}}$ if and only if $\phi(A)^{\frac{\alpha}{z}} \leq \phi(B)^{\frac{\alpha}{z}}$. Indeed,

$$A^{\frac{\alpha}{z}} \leq B^{\frac{\alpha}{z}} \Leftrightarrow \tau\text{-}Q_{\alpha,z}(A||X) \leq \tau\text{-}Q_{\alpha,z}(B||X), \;\; X \in \mathcal{A}^{++}$$
$$\Leftrightarrow \omega\text{-}Q_{\alpha,z}(\phi(A)||\phi(X)) \leq \omega\text{-}Q_{\alpha,z}(\phi(B)||\phi(X)), \;\; X \in \mathcal{A}^{++} \Leftrightarrow \phi(A)^{\frac{\alpha}{z}} \leq \phi(B)^{\frac{\alpha}{z}}.$$

Positive homogeneity: for given $A \in \mathcal{A}^{++}$ and $t > 0$, and for arbitrary $X \in \mathcal{A}^{++}$ we compute

$$\begin{aligned} \omega\text{-}Q_{\alpha,z}(\phi(tA)||\phi(X)) &= \tau\text{-}Q_{\alpha,z}(tA||X) = t^{\alpha}\tau\text{-}Q_{\alpha,z}(A||X) \\ &= t^{\alpha}\omega\text{-}Q_{\alpha,z}(\phi(A)||\phi(X)) = \omega\text{-}Q_{\alpha,z}(t\phi(A)||\phi(X)). \end{aligned}$$

We conclude that $\phi(tA) = t\phi(A)$.

Denote by ψ the map $A \mapsto \phi(A^{\frac{z}{\alpha}})^{\frac{\alpha}{z}}$. It is a homogeneous order isomorphism.

We obtain that ψ is of the form $\psi(A) = CJ(A)C$, $A \in \mathcal{A}^{++}$, where $J : \mathcal{A} \to \mathcal{B}$ is a Jordan *-isomorphism and $C \in \mathcal{B}^{++}$.

Concerning the original transformation ϕ, this means that

$$\phi(A) = \psi\left(A^{\frac{\alpha}{z}}\right)^{\frac{z}{\alpha}} = \left(CJ(A^{\frac{\alpha}{z}})C\right)^{\frac{z}{\alpha}}, \quad A \in \mathcal{A}^{++}.$$

If we write $B = A$ in the original preserver identity, we get $\omega(\phi(A)) = \tau(A)$, $A \in \mathcal{A}^{++}$. Therefore, we have

$$\omega\left((CJ(A^{\frac{\alpha}{z}})C)^{\frac{z}{\alpha}}\right) = \tau(A), \quad A \in \mathcal{A}^{++}.$$

This implies (see Lemma 9) that $C \in \mathcal{B}$ is a central element.
Easy to complete the proof.

II. In the case where $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ is a surjective map with the property that

$$\omega\text{-}Q_\alpha^{max}(\phi(A)||\phi(B)) = \tau\text{-}Q_\alpha^{max}(A||B), \quad A, B \in \mathcal{A}^{++}, \tag{16}$$

the proof is quite similar.

III. Let now $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ be a surjective map which satisfies

$$\omega\text{-}Q_\alpha^{mo}(\phi(A)||\phi(B)) = \tau\text{-}Q_\alpha^{mo}(A||B), \quad A, B \in \mathcal{A}^{++}. \tag{17}$$

Here the situation is different. Using the corresponding order characterization (Lemma 8) we deduce that ϕ has the following property: for any $A, B \in \mathcal{A}^{++}$ we have

$$\log A \leq \log B \iff \log \phi(A) \leq \log \phi(B).$$

We can also obtain that ϕ is positive homogeneous.

Let us now consider the map $\psi : T \mapsto \log \phi(\exp(T))$ on $\mathcal{A}_s$. It is apparent that ψ is an order isomorphism between the spaces $\mathcal{A}_s$ and $\mathcal{B}_s$ of self-adjoint elements of $\mathcal{A}$ and $\mathcal{B}$, respectively. Moreover, because of the homogeneity of ϕ, we calculate

$$\psi(T+tI) = \log \phi(e^t \exp(T)) = \log e^t \phi(\exp(T)) = tI + \psi(T), \quad T \in \mathcal{A}_s, t \in \mathbb{R}.$$

Therefore, for any $T, S \in \mathcal{A}_s$ and real number t,

$$T-S \leq tI \Leftrightarrow T \leq S+tI \Leftrightarrow \psi(T) \leq \psi(S+tI) = \psi(S)+tI \Leftrightarrow \psi(T)-\psi(S) \leq tI.$$

Apparently,

$$\|X\| = \max\{\min\{t \in \mathbb{R} : X \leq tI\}, \min\{t \in \mathbb{R} : -X \leq tI\}\}. \tag{18}$$

We obtain that ψ satisfies

$$\|\psi(T) - \psi(S)\| = \|T - S\|, \quad T, S \in \mathcal{A}_s,$$

i.e., ψ is a surjective isometry between the normed real linear spaces $\mathcal{A}_s$ and $\mathcal{B}_s$.

Mazur-Ulam theorem asserts that any surjective isometry between normed real linear spaces is affine and hence it is a surjective linear isometry followed by a translation. The surjective linear isometries between the spaces $\mathcal{A}_s, \mathcal{B}_s$ were completely determined by Kadison. Applying his result we have a central symmetry $C \in \mathcal{B}$, a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$ and an element X_0 in $\mathcal{B}_s$ such that

$$\log(\phi(\exp(T))) = \psi(T) = CJ(T) + X_0, \quad T \in \mathcal{A}_s.$$

It follows that ϕ is necessarily of the form

$$\phi(A) = \exp(CJ(\log A) + X_0), \quad A \in \mathcal{A}^{++}. \tag{19}$$

What remains is to verify that C is the identity (easy) and that X_0 is a central element (an application of a former lemma on centrality).

Similar results concerning other fundamental concepts of quantum relative entropy: the Umegaki relative entropy and the Belavkin-Staszewski relative entropy.

For any C^*-algebra $\mathcal{A}$ with faithful trace τ, the Umegaki relative entropy on $\mathcal{D}_\tau^{-1}(\mathcal{A})$ is defined by

$$S_U^\tau(A||B) = \tau\left(A(\log A - \log B)\right) \tag{20}$$

for all $A, B \in \mathcal{D}_\tau^{-1}(\mathcal{A})$, while the Belavkin-Staszewski relative entropy is defined by

$$S_{BS}^\tau(A||B) = \tau\left(A\log(A^{1/2}B^{-1}A^{1/2})\right) \tag{21}$$

for all $A, B \in \mathcal{D}_\tau^{-1}(\mathcal{A})$.

Connection to the quantum Rényi relative entropies:
In the finite dimensional setting, where $\mathcal{A}$ is the algebra of all $n \times n$ complex matrices and τ is the usual trace,
the conventional Rényi relative entropy, the minimal Rényi relative entropy, even the general $\alpha - z$-Rényi relative entropy and also the Mosonyi-Ogawa version of quantum Rényi relative entropy tend to the Umegaki relative entropy as $\alpha \to 1$.
The limiting case of the maximal Rényi relative entropy is the Belavkin-Staszewski relative entropy.

The structure of the corresponding preservers is the same as before.

Theorem 11

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with faithful traces τ, ω, respectively, and let $\phi : \mathcal{D}_\tau^{-1}(\mathcal{A}) \to \mathcal{D}_{\tau'}^{-1}(\mathcal{B})$ be a surjective map. Then ϕ preserves the Umegaki relative entropy, i.e., it satisfies

$$S_U^\omega(\phi(A)||\phi(B)) = S_U^\tau(A||B), \quad A, B \in \mathcal{D}_\tau^{-1}(\mathcal{A})$$

*if and only if there is a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$ and an element $C \in \mathcal{B}^{++}$ central in the algebra $\mathcal{A}$ such that $\phi(A) = CJ(A)$, $A \in \mathcal{D}_\tau^{-1}(\mathcal{A})$ and $\omega(CJ(X)) = \tau(X)$ holds for all $X \in \mathcal{A}$.*

Theorem 12

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with faithful traces τ, ω, respectively, and $\phi : \mathcal{D}_\tau^{-1}(\mathcal{A}) \to \mathcal{D}_{\tau'}^{-1}(\mathcal{B})$ be a surjective map. Then ϕ preserves the Belavkin-Staszewski relative entropy, i.e., it satisfies

$$S_{BS}^{\omega}(\phi(A)||\phi(B)) = S_{BS}^{\tau}(A||B), \quad A, B \in \mathcal{D}_\tau^{-1}(\mathcal{A}) \tag{22}$$

*if and only if there is a Jordan *-isomorphism $J : \mathcal{A} \to \mathcal{B}$ and an element $C \in \mathcal{B}^{++}$ central in the algebra $\mathcal{B}$ such that $\phi(A) = CJ(A)$, $A \in \mathcal{D}_\tau^{-1}(\mathcal{A})$ and $\omega(CJ(X)) = \tau(X)$ holds for all $X \in \mathcal{A}$.*

Here we also need characterizations of the order expressed by Umegaki and Belavkin-Staszewski relative entropies:

Lemma 13

Let $\mathcal{A}$ be a C^*-algebra with a faithful trace τ. Select $A, B \in \mathcal{A}^{++}$. We have $\log A \leq \log B$ if and only if $S_U^\tau(X||B) \leq S_U^\tau(X||A)$ holds for all $X \in \mathcal{A}^{++}$.

Lemma 14

Let $\mathcal{A}$ be a C^*-algebra with a faithful trace τ. For any $A, B \in \mathcal{A}^{++}$, we have $A \leq B$ if and only if $S_{BS}^\tau(X||B) \leq S_{BS}^\tau(X||A)$ holds for all $X \in \mathcal{A}^{++}$.

Similar further results: description of Bures-Wasserstein and Hellinger isometries between positive definite cones or density spaces of C^*-algebras, fidelity preserving maps, etc.

Next question:

How different the above defined quantum Rényi relative entropies are?
Can one of them transform by any map to another one?

Answer: only in the case of commutative algebras, when all of them are trivially the same.

Theorem 15

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with faithful traces τ, ω, respectively, and let $\phi : \mathcal{D}_\tau^{-1}(\mathcal{A}) \to \mathcal{D}_{\tau'}^{-1}(\mathcal{B})$ be a surjective map. Assume that ϕ satisfies

$$\omega\text{-}D_\alpha^{mo}((\phi(A)||\phi(B)) = \tau\text{-}D_{\alpha,z}(A||B), \quad A, B \in \mathcal{D}_\tau^{-1}(\mathcal{A}).$$

Then the algebras $\mathcal{A}, \mathcal{B}$ are necessarily commutative in which case all the considered types of quantum Rényi relative entropy coincide.
*Then Corollary 4 applies and provides the form of ϕ, in which the corresponding Jordan *-isomorphism is of course necessarily an algebra *-isomorphism.*
Analogous assertions hold for all other pairs of different quantum Rényi relative entropies listed in (4)-(8) and for every positive number α different from 1 with the only restriction that concerning the quantity in (7) we need to assume that $\alpha \leq 2$.

Proofs: We argue by contradiction, assume the existence of such a transformation. Again, we use the order preserving properties but not only them. The details are more complicated than in the cases of the symmetries what we have considered in the first part of the talk.
For example, at some point we need the following result which might be interesting on its own right: no proper dilation with respect to the Thompson metric in noncommutative algebras.

Theorem 16

If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and there is a surjective map $\phi : \mathcal{A}^{++} \to \mathcal{B}^{++}$ such that

$$d_T(\phi(A), \phi(B)) = \gamma d_T(A, B), \quad A, B \in \mathcal{A}^{++} \tag{23}$$

holds with some positive real number γ different from 1, then the algebras $\mathcal{A}, \mathcal{B}$ are necessarily commutative.

As for transformations between the Umegaki and the Belavkin-Staszewski relative entropies, we have the following analogue of Theorem 15.

Theorem 17

Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with faithful traces τ, ω, respectively, and let $\phi : \mathcal{D}_\tau^{-1}(\mathcal{A}) \to \mathcal{D}_{\tau'}^{-1}(\mathcal{B})$ be a surjective map which satisfies

$$S_U^\omega(\phi(A)||\phi(B)) = S_{BS}^\tau(A||B), \quad A, B \in \mathcal{D}_\tau^{-1}(\mathcal{A}).$$

*Then the algebras $\mathcal{A}, \mathcal{B}$ are necessarily commutative in which case the Umegaki and the Belavkin-Staszewski relative entropies coincide. Hence Theorem 11 or 12 applies and provides the form of ϕ, in which the corresponding Jordan *-isomorphism is of course necessarily an algebra *-isomorphism.*

Thank you very much for your kind attention!