Finite Rank Perturbations

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at OTOA in Bangalore
December 2018

This talk is based on joint work with Sergei Treil.
Thanks also to NSF.

Setting and definitions

Given operator A, what can we say about the spectral properties of

$$A + B \quad \text{for} \quad B \in \text{Class } X?$$

- Classically Class $X = \{\text{trace cl.}\}$, $\{\text{Hilb.–Schmidt}\}$, $\{\text{comp.}\}$.
- Here: A, B self-adj. on separable $\mathcal{H}$, Class $X = \{\text{finite rk}\}$.

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Definition

Through $A_\gamma = A + \gamma(\,\cdot\,, \varphi)\varphi$, parameter $\gamma \in \mathbb{R}$ realizes all *self-adjoint rank one perturbations* (of a given self-adjoint operator A) in the direction of a cyclic φ (WLOG).

Definition

Through $A_\Gamma = A + \mathbf{B}\Gamma\mathbf{B}^*$, the symmetric $d \times d$ matrices Γ parametrize all *self-adjoint finite rank perturbations* with range contained in that of $\mathbf{B}$. WLOG: Range $\mathbf{B}$ is a cyclic subspace and $\mathbf{B} : \mathbb{C}^d \to \mathcal{H}$ left-invertible on its range.

Cyclicity examples on $\mathbb{R}^2$ (recall $A_\Gamma = A + \mathbf{B}\Gamma\mathbf{B}^*$)

- $A_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \gamma(\,\cdot\,, e_1)e_1$. Here e_1 is not cyclic.
- $A_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \gamma(\,\cdot\,, e_1 + e_2)(e_1 + e_2)$.
  - $e_1 + e_2$ is cyclic for A_γ for all $\gamma \in \mathbb{R}$.
- $A_{\gamma_1,\gamma_2} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \gamma_1(\,\cdot\,, e_1)e_1 + \gamma_2(\,\cdot\,, e_2)e_2$.
  - A_{γ_1,γ_2} cannot be written as rank one perturbation.
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  - $e_1 + e_2$ is cyclic iff $\gamma_1 \neq \gamma_2 - 2$.
  - Cyclicity of A does not imply that of A_Γ; because for $\gamma_1 = \gamma_2 - 2$, above operator $A_{\gamma_1,\gamma_2} = (1 + \gamma_1)\mathbf{I}$ is not cyclic.

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On finite dimensional space, this theory reduces to finding EVA through diagonalization $UA_\gamma = DU$.

Scalar measure and decomposition

Theorem (Scalar Spectral Theorem)

Let A be a self-adjoint operator on Hilbert space $\mathcal{H}$ with (cyclic) vector φ. Then there exists a unique measure $\mu = \mu^\varphi$ such that

$$\big((A - z\mathbf{I})^{-1}\varphi, \varphi\big)_{\mathcal{H}} = \big((M_t - z\mathbf{I})^{-1}\mathbf{1}, \mathbf{1}\big)_{L^2(\mu)} = \int \frac{d\mu(t)}{t - z} =: F(z)$$

for $z \in \mathbb{C}\backslash\mathbb{R}$. Namely, $(A$ on $\mathcal{H}) \sim (M_t$ on $L^2(\mu))$.

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- μ contains all the spectral information of operator A.
- EVA λ of A is reflected in point mass at λ, i.e. $\mu\{\lambda\} > 0$.
- Lebesgue decompose the spectral measure $d\mu = d\mu_{\mathrm{ac}} + d\mu_{\mathrm{s}}$.
- Further decompose $d\mu_{\mathrm{s}} = d\mu_{\mathrm{p}} + d\mu_{\mathrm{sc}}$.
- Through $A \sim M_t$ decompose operator $A = A_{\mathrm{ac}} \oplus A_{\mathrm{p}} \oplus A_{\mathrm{sc}}$.

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- Further decompose $d\mu_{\mathrm{s}} = d\mu_{\mathrm{p}} + d\mu_{\mathrm{sc}}$.
- Through $A \sim M_t$ decompose operator $A = A_{\mathrm{ac}} \oplus A_{\mathrm{p}} \oplus A_{\mathrm{sc}}$.
- The spectrum $\sigma := \operatorname{supp} \mu$.
- Alternatively decompose $A = A_{\mathrm{ess}} \oplus A_{\mathrm{d}}$ and $\sigma = \sigma_{\mathrm{ess}} \dot\cup \sigma_{\mathrm{d}}$.

Perturbation theory (A, $T = A + B$ self-adjoint)

- $A \sim T$(Mod compact operators) means $UA = TU + K$ for some unitary U and compact K.

Theorem (Weyl–vonNeuman early 1900's)

$A \sim T$(*Mod compact operators*) $\quad \Leftrightarrow \quad \sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(T)$.

Theorem (Kato–Rosenblum 1950's, Carey–Pincus 1976)

$A \sim T$(*Mod trace class*) $\quad \Leftrightarrow \quad A_{\mathrm{ac}} \sim T_{\mathrm{ac}}$, *conditions.*

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Theorem (Aronszajn–Donoghue Theory 1970-80's)

Spectral type is not stable under rank one perturbations. The singular parts of A and A_γ are mutually singular. (next slide)

Theorem (Poltoratski 2000)

Conditions on purely singular operators $\Rightarrow A \sim T$(Mod rank 1).

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Barry Simon: "The cynic might feel that I have finally sunk to my proper level [...] to rank one perturbations – maybe something so easy that I can say something useful! We'll see even this is hard and exceedingly rich."

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When $\gamma \neq 0$, the sets

$$S_\gamma = \left\{ x \in \mathbb{R} \,\middle|\, \lim_{y \searrow 0} F(x+iy) = -1/\gamma; G(x) = \infty \right\},$$

$$P_\gamma = \left\{ x \in \mathbb{R} \,\middle|\, \lim_{y \searrow 0} F(x+iy) = -1/\gamma; G(x) < \infty \right\},$$

$$C = \operatorname{clos}\left\{ x \in \mathbb{R} \,\middle|\, \lim_{y \searrow 0} \operatorname{Im} F(x+iy) \neq 0 \right\}$$

contain spectral information of A_γ as follows:

(i) *Set P_γ is the set of eigenvalues, and set C (S_γ) is a carrier for the absolutely (singular) continuous measure, respectively.*

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- Main tool: Aronszajn–Krein formula $F_\gamma = \frac{F}{1+\gamma F}$.
- Literature provides finer results and pathological examples.

Origins and applications of rank one perturbations

- Describe all self-adjoint extensions of a symmetric operator with deficiency indices $(1, 1)$.
- Differential operators with changing boundary conditions:
  - Sturm–Liouville operators (Weyl 1910),
  - Half-line Schrödinger operator $Au = -\frac{d^2}{dx^2}u + Vu$,
  - Maybe soon PDEs.
- Anderson-type Hamiltonian

$$A_\omega = A + \sum_{m\in\mathbb{N}} \omega_m(\,\cdot\,, \varphi_m)\varphi_m$$

for orthonormal φ_m and i.i.d. random ω_m, $\omega = (\omega_1, \omega_2, \ldots)$. For example, the discrete random Schrödinger operator.

Rank one perturbations and analysis

- Nehari interpolation problem
- Holomorphic composition operators
- Rigid functions
- Functional models (Nagy–Foiaș, deBranges–Rovnyak, Nikolski–Vasyunin)
- Two weight problem for Hilbert/Cauchy transform
- Existence of the limit in the Julia–Carathéodory quotient
- Carlesson embedding

What are finite rank perturbations related to?

- Describe all self-adjoint extensions of a symmetric operator with finite deficiency indicees (d, d).
- Second order differential operators with both endpoints limit circle.
- Higher order differential operators.
- Functional models with matrix-valued characteristic functions (Nagy–Foiaș, deBranges–Rovnyak, Nikolski–Vasyunin).
- Ex. of the limit in the (matrix-valued) Julia–Carath. quotient.
- Two weight problem for Hilbert/Cauchy transform with matrix-valued weights.

Subset of interested people

Unitary rank one perturbations or their corresponding model spaces were studied by Aleksandrov, Ball, Clark, Douglas–Shapiro–Shields, Kapustin, Poltoratski, Ross, Sarason, etc.

A self-adjoint setting was studied by Albeverio–Kurasov, Aronszajn–Donoghue, delRio, Kato-Rosenblum, Poltoratski, Simon, etc.

Finite rank generalizations occur in literature by Albeverio–Kurasov (extension theory), Gesztesy et al., Kapustin–Poltoratski (purely singular spectra), Martin.

Baranov has extended some aspects to rank one perturbations of normal operators.

Matrix-valued spectral measures

With $b_k = \mathbf{B}\mathbf{e}_k$, for $k = 1, 2, \ldots, d$, consider *(singular) form bounded perturbations*, that means that for each k we have

$$\|(1 + |A|)^{-1/2} b_k\|_{\mathcal{H}} < \infty \qquad \text{where } |A| = (A^*A)^{1/2}.$$

Theorem (Matrix-valued Spectral Theorem)

Let A be a self-adjoint on $\mathcal{H}$ with cyclic set $\{b_k\}$. Then there is a unique matrix-valued measure $\mathbf{M}$ so that

$$\mathbf{B}^*(A - z\mathbf{I})^{-1}\mathbf{B} = \int \frac{d\mathbf{M}(t)}{t - z} \qquad \textit{for } z \in \mathbb{C}\backslash\mathbb{R};$$

i.e. $A \sim M_t$ *on* $L^2(\mathbf{M})$ *with* $\|f\|^2_{L^2(\mathbf{M})} = \int ([d\mathbf{M}(t)]f(t), f(t))_{\mathbb{C}^d}$.

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We associate scalar spectral measure $\mu := \operatorname{tr} \mathbf{M}$. Then $d\mathbf{M} = W d\mu$ with $W = B^*B$, $B(t) = (Ub_1(t), Ub_2(t), \ldots)$, and the vector-valued integral

$$\int [d\mathbf{M}] f = \int W(t) f(t) d\mu(t).$$

Spectral Measure of $A_\Gamma = A + \mathbf{B}\Gamma\mathbf{B}^*$ and decomposition

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  defines the family $\{\mathbf{M}_\Gamma\}$ of spectral measures of A_Γ.
- With $\mu_\Gamma := \mathrm{tr}\,\mathbf{M}_\Gamma$ and $W_\Gamma = B_\Gamma^* B_\Gamma$ we have

$$d\mathbf{M}_\Gamma = W_\Gamma d\mu_\Gamma.$$

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- Lebesgue decomp. $d\mu = w dx + d\mu_{\mathrm{s}}$ yields:

$$d\mathbf{M}(x) = d\mathbf{M}_{\mathrm{ac}}(x) + d\mathbf{M}_{\mathrm{s}}(x).$$

- Aronszajn–Krein-type: $F_\Gamma = (\mathbf{I} + F\Gamma)^{-1}F = F(\mathbf{I} + \Gamma F)^{-1}$.

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Theorem (Wave operators)

The wave operators exist;
i.e. defining $\mathcal{W}^\Gamma(\tau) := e^{i\tau A_\Gamma} e^{-i\tau A} P_{\mathrm{ac}}$, where P_{ac} is the orth. proj. onto the absolutely continuous part of A, the strong limit s-lim$_{\tau\to\pm\infty}\mathcal{W}^\Gamma(\tau)$ exists.

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Idea of proof for wave operators: For any $f \in L^2(\mathbf{M}_{\mathrm{ac}})$ we have

$$\text{s-lim}_{\tau\to\pm\infty} V_\Gamma P_{\mathrm{ac}}^{A_\Gamma} \mathcal{W}^\Gamma(\tau) f = (\mathbf{I} + \Gamma F_\pm) f.$$

Spectral representation of A_Γ

Consider the *spectral representation* of $A_\Gamma = M_t + \mathbf{B}\Gamma\mathbf{B}^*$, i.e. unitary $V_\Gamma : L^2(\mathbf{M}) \to L^2(\mathbf{M}^\Gamma)$ with $V_\Gamma A_\Gamma = M_s V_\Gamma$.

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Theorem (L.–Treil)

The spectral representation takes the form

$$(V_\Gamma h\mathbf{e})(s) = h(s)\mathbf{e} - \Gamma \int \frac{h(t) - h(s)}{t - s} [d\mathbf{M}(t)]\mathbf{e}$$

for $\mathbf{e} \in \mathbb{C}^d$ *and compactly supported* $h \in C^1(\mathbb{R})$.

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For a matrix-valued measure $\mathbf{M} = W\mu$ and $\varepsilon > 0$, define operator $T^{\mathbf{M}}_{\pm\varepsilon} : L^2(\mathbf{M}) \to L^2(\Gamma\mathbf{M}^\Gamma\Gamma)$ by

$$T^{\mathbf{M}}_{\pm\varepsilon} f(s) = \int [d\mathbf{M}(t)] \frac{f(t)}{s - t \pm i\varepsilon}.$$

Theorem (L.–Treil)

Operators $T^{\mathbf{M}}_{\pm\varepsilon} : L^2(\mathbf{M}) \to L^2(\Gamma\mathbf{M}^\Gamma\Gamma)$ *are (uniformly in* ε*) bounded with norm at most* 2.

Vector mutual singularity of singular parts

Definition

Matrix-valued measures $\mathbf{M} = W\mu$ and $\mathbf{N} = V\nu$ are *vector mutually singular* $(\mathbf{M} \perp \mathbf{N})$ if one can extent W and V so that

$$\operatorname{Ran} W(t) \perp \operatorname{Ran} V(t) \qquad \mu\text{-a.e. and } \nu\text{-a.e.}$$

Theorem (L.–Treil)

Singular parts of the matrix-valued measures M and M^{Γ} satisfy

$$\mathbf{M}_{\mathrm{s}} \perp \Gamma \mathbf{M}_{\mathrm{s}}^{\Gamma} \Gamma \qquad \textit{and} \qquad \mathbf{M}_{\mathrm{s}}^{\Gamma} \perp \Gamma \mathbf{M}_{\mathrm{s}} \Gamma.$$

The proof uses uniform boundedness of the spectral representation and a matrix A_2 condition.

Aleksandrov Spectral Averaging

Theorem (L.–Treil)

Let Γ_0 *be a self-adjoint and* Γ_1 *be a positive definite* $d \times d$ *matrix. Consider scalar-valued Borel measurable* $f \in L^1(\mathbb{R})$*. We have*

$$\iint f(x) d\mathbf{M}_{\Gamma_0 + t\Gamma_1}(x) dt = \Gamma_1^{-1} \int f(x) dx.$$

In particular, for any Borel set B *with zero Lebesgue measure* $\mathbf{M}_{\Gamma_0 + t\Gamma_1}(B) = \mathbf{0}$ *for Lebesgue a.e.* $t \in \mathbb{R}$*.*

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- We can also show that the scalar-valued spectral measures $\mu_\Gamma(B) = 0$ for Lebesgue almost all Γ in a "cylinder".
- For an arbitrary singular Radon measure ν on $\mathbb{R}$, $\nu \perp (\mu_{\Gamma_0 + t\Gamma_1})_{\mathrm{s}}$ for all except maybe countably many $t \in \mathbb{R}$.

Summary

- Kato–Rosenblum simple proof and existence of wave operators
- Vector mutual singularity of matrix-valued spectral measures
- Aleksandrov spectral averaging yields some mutual singularity also of scalar-valued spectral measures
- Proofs and results from rank one perturbation theory needed to be changed