The metric space of norms, and the normed space of metrics

Apoorva Khare

Indian Institute of Science
and
Analysis & Probability Research Group, Bangalore

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Norms on finite-dimensional Banach spaces

Definition: A *norm* on a Banach space $\mathbb{B}$ over $\mathbb{R}$ or $\mathbb{C}$ is a map $\| \cdot \| : \mathbb{B} \to [0, \infty)$ that vanishes only at 0, is sub-additive, and is homogeneous:

$$\|\alpha v\| = |\alpha| \cdot \|v\|, \qquad \forall v \in \mathbb{B} \text{ and scalars } \alpha.$$

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Folklore: Fix $k \geqslant 1$. All norms on $\mathbb{R}^k$ are topologically equivalent – i.e., Lipschitz equivalent: there exist $0 < m \leqslant M$ such that

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This equivalence can be quantified into a pseudometric: the *Banach–Mazur distance*. Namely, if $U := (\mathbb{R}^k, \|\cdot\|)$ and $U' := (\mathbb{R}^k, \|\cdot\|')$, then

$$d_{BM}(\|\cdot\|\ ,\ \ \|\cdot\|') = d_{BM}(U, U') = \log\inf\{\|T\| \cdot \|T^{-1}\| : T \in GL_{\mathbb{R}}(U, U')\}.$$

Thus, $d_{BM}(U, U') = 0$ if and only if there exists a linear isometry $T : U \to U'$.

The Banach–Mazur compactum

Definition: Quotient the space of norms on $\mathbb{R}^k$ by the equivalence relation

$$U \sim U' \iff d_{BM}(U, U') = 0.$$

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Properties of $\mathcal{Q}(\mathbb{R}^k)$:

1. $\mathcal{Q}(\mathbb{R}^k)$ is a compact, connected metric space.
2. Examples of distances in $\mathcal{Q}(\mathbb{R}^k)$: if $p, q \in [1, 2]$ or $p, q \in [2, \infty]$ then

$$d_{BM}(\| \cdot \|_p, \ \| \cdot \|_q) = \log(k) \cdot |1/p - 1/q| \, .$$

However, this does not hold if $p < 2 < q$; for instance, if $k = 2$ then $d_{BM}(\| \cdot \|_1, \| \cdot \|_\infty) = 0$.

A new metric on norms

Recall: two norms are equivalent if $\exists 0 < m \leqslant M$ such that

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for the tightest possible m, M.

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The following is a (different) pseudometric on the space of norms:

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Questions:

1. Is $\mathcal{Q}_{coarse}(\mathbb{R}^k)$ related to the Banach–Mazur compactum $\mathcal{Q}(\mathbb{R}^k)$?
2. What are the properties of $\mathcal{Q}_{coarse}(\mathbb{R}^k)$?

Coarse Banach–Mazur space: properties

Lemma (Distances between p-norms)

$$d_{coarse}(\| \cdot \|_p \ , \ \ \| \cdot \|_q) = \log(k) \left| 1/p - 1/q \right| , \qquad \forall p, q \in [1, \infty].$$

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Proof sketch: If $1 \leqslant p < q < \infty$ then Hölder's inequality implies:

$$k^{-1/p} \|v\|_p \leqslant k^{-1/q} \|v\|_q, \qquad \forall v \in \mathbb{R}^k,$$

with equality if $v = (c, \ldots, c)^T$.

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More generally:

Lemma

The Banach–Mazur compactum $\mathcal{Q}(\mathbb{R}^k)$ is a topological quotient of $\mathcal{Q}_{coarse}(\mathbb{R}^k)$.

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More generally:

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The Banach–Mazur compactum $\mathcal{Q}(\mathbb{R}^k)$ is a topological quotient of $\mathcal{Q}_{coarse}(\mathbb{R}^k)$.

Proof: If two norms are proportional (hence the same point in $\mathcal{Q}_{coarse}(\mathbb{R}^k)$), then $\|T\| \cdot \|T^{-1}\| = 1$ for $T = \mathrm{Id}$. So they have distance zero in $\mathcal{Q}(\mathbb{R}^k)$. □

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Theorem (K., 2018)

1. *For each integer $k \geqslant 1$, the projective space $\mathcal{Q}_{coarse}(\mathbb{R}^k)$ of norms on $\mathbb{R}^k$ is a complete, path-connected metric subspace of a real Banach space.*

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To understand this norm, the metric embedding is done as follows:

- Fix $X = S^{k-1}$, the unit sphere, and send a norm N on $\mathbb{R}^k$ to $\log N|_X : X \to \mathbb{R}$. Thus $\mathcal{Q}_{coarse}(\mathbb{R}^k) \hookrightarrow C(X, \mathbb{R}) \dots$ modulo what?

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- *Unboundedness:* Define the norms $N_{q,j}(v) := \|v\|_2 + q|v_j|$, for $q \geqslant 0,\ j = 1, \dots, k$. The map : $N_{q,j} \mapsto \log(1 + q)\mathbf{e}_j$ is an isometry onto the non-negative coordinate axes in $(\mathbb{R}^k, \| \cdot \|_1)$.

Distortion

The same construction applies to metrics on the finite set

$$[n] := \{1, \dots, n\}, \qquad n \geqslant 1.$$

Given metrics $\rho, \rho' : [n] \times [n] \to [0, \infty)$, define

$$d_{coarse}(\rho, \rho') \ := \ \log \max_{j \neq k \in [n]} \frac{\rho'(j,k)}{\rho(j,k)} \ - \ \log \min_{j \neq k \in [n]} \frac{\rho'(j,k)}{\rho(j,k)}.$$

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Remark 2: This metric is not comparable to another well-known metric on 'metrics on $[n]$': the *Gromov–Hausdorff metric*.

The coarse Banach–Mazur space of metrics

Theorem (K., 2018)

Fix an integer $n \geqslant 2$.

1. *The map d_{coarse} is a pseudometric on the space of metrics on $[n]$, with equivalence classes precisely consisting of proportional metrics.*
2. *The quotient metric space $\mathcal{Q}_{coarse}([n])$ is a complete, path-connected, metric subspace of the Banach space $\mathbb{R}^{\binom{[n]}{2}}/\sim\ = C(\binom{[n]}{2}, \mathbb{R})/\sim$ with the diameter norm.*

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Thus the (proportionality classes of) metrics on $[n]$ sit in a normed space.

Connection to Lipschitz distance

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These constructions and facts are completely parallel to the story for norms on $\mathbb{R}^k$. How to understand these parallels (common framework)?

More general frameworks

The above settings can be generalized. For example:

Theorem (K., 2018)

Let $\mathbb{B}$ be a Banach space over $\mathbb{R}$, and let $X \subset \mathbb{B} \setminus \{0\}$ be such that every ray $\mathbb{R}^+ v,\ 0 \neq v \in \mathbb{B}$ intersects X.
Let $\mathcal{N}_X$ consist of all norms N on $\mathbb{B}$ such that $N|_X$ is bounded away from $0, \infty$. Then the proportionality classes in $\mathcal{N}_X$ form a complete, path-connected metric subspace of the Banach space $C(X, \mathbb{R})/\sim$ with the diameter norm.

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Theorem

For any set X of size at least 3, the equivalence classes (again by scaling) of metrics ρ on X bounded away from $0, \infty$ outside the diagonal – i.e., such that

$$0 < \inf_{x \neq x' \in X} \rho(x, x') \leqslant \sup_{x \neq x' \in X} \rho(x, x') < \infty$$

form a complete, path-connected unbounded metric subspace of the Banach space $C(\binom{X}{2}, \mathbb{R})/\sim$. Here $\binom{X}{2}$ comprises the pairs of elements in X.

General framework: symmetry group

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- The key extra ingredient is a *symmetry group* G of the 'structured set' X ($G =$ invertible linear maps; permutations). Now we define the G*-distance* between two metrics $\rho, \rho' \in \mathcal{C}$ to be:

$$d_G(\rho, \rho') := \log \inf_{g \in G} \{\|g\| \cdot \|g^{-1}\|\},$$

where one defines:

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General framework: symmetry group

What we are working with in **all** of the above examples is:

- A set X – e.g. a k-dimensional $\mathbb{R}$-vector space; or a set of size n.
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In both of these settings, $\mathcal{Q}$ is clearly a quotient of $\mathcal{Q}_{coarse}$.

Concluding questions; References

1. Find 'more familiar' (geometric) models for the spaces $\mathcal{Q}_{coarse}(\mathbb{R}^k)$, $\mathcal{Q}_{coarse}([n])$.
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[1] The non-compact normed space of norms on a finite-dimensional Banach space
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