A glimpse at linear dynamics

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Recent Advances in Operator Theory and Operator Algebras
Bangalore, December 17, 2018

1. Linear dynamics
2. Chaos vs. frequent hypercyclicity
3. Further recent advances

Linear dynamics

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Good, short introduction: Aneesh M (talk)

Three basic notions

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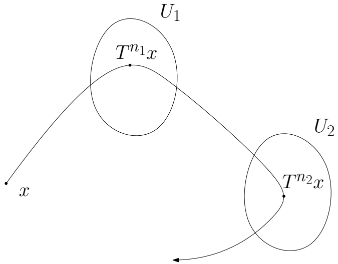

Each such vector x is called a hypercyclic vector for T.

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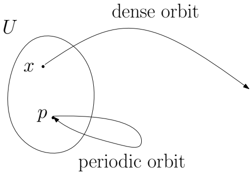

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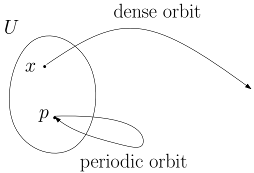

This implies that the dynamical system has sensitive dependence on initial conditions.

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where $\underline{\text{dens}}$ is the lower density of a set.

In other words, x is frequently hypercyclic for T if, for any open set $U \neq \varnothing$,

$$\exists n_k \text{ with } n_k = O(k) \text{ such that } T^{n_k} x \in U, k \geq 1.$$

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Let $X = \ell^p$, $1 \leq p < \infty$, or c_0. Let $w = (w_n)_n$ be a bounded sequence with $w_n \neq 0$, $n \geq 1$. Then a weighted backward shift is given by

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In particular, the multiple of the backward shift B

$$T = \lambda B : (x_n)_n \to \lambda(x_{n+1})_n, \quad |\lambda| > 1$$

is

- hypercyclic (Rolewicz 1969),
- chaotic (Godefroy-Shapiro 1991), and
- frequently hypercyclic (Bayart-Grivaux 2006).

The characterization of hypercyclic and chaotic weighted shifts is well-known (Salas 1995, GE 2000):

$$B_w \text{ hypercyclic } \iff \sup_n \prod_{k=1}^n |w_k| = \infty.$$

$$B_w \text{ chaotic } \iff \begin{cases} \sum_n \frac{1}{\prod_{k=1}^n |w_k|^p} < \infty, & \text{if } X = \ell^p, \\ \lim_n \prod_{k=1}^n |w_k| = \infty, & \text{if } X = c_0. \end{cases}$$

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The characterization of the frequently hypercyclic weighted backward shifts had been an open problem until recently ($\longrightarrow$ later).

Another well-known hypercyclic operator (MacLane 1952) is the differentiation operator

$$D : H(\mathbb{C}) \to H(\mathbb{C}), \quad f \to f'.$$

Here, $H(\mathbb{C})$ is the Fréchet space of entire functions on $\mathbb{C}$, endowed with the compact-open topology.

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Note that D is also a weighted shift:

$$D\Big(\sum_n a_n z^n\Big) = \sum_n (n+1)a_{n+1} z^n.$$

Chaos vs. frequent hypercyclicity

How can the orbit of a vector x be frequently hypercyclic?
Recall that we need that, for any open set $U \neq \varnothing$,

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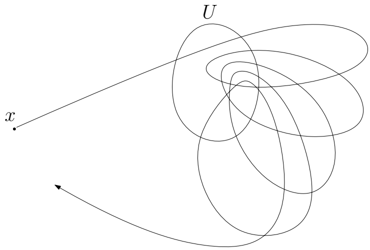

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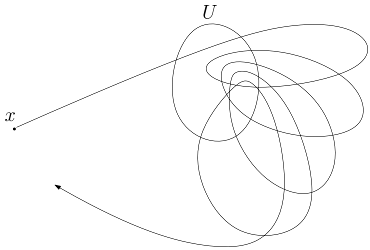

This looks as if in U one has a periodic point.

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Theorem (Bayart-Grivaux 2007)

There is a weighted backward shift B_w on c_0 that is frequently hypercyclic but that does not have any non-trivial periodic points. It is therefore not chaotic.

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Based on these conditions, Bonilla-GE (2018) have given a simpler construction of a frequently hypercyclic shift on c_0 that has no non-trivial periodic points.

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Let B_w be a weighted backward shift on ℓ^p, $1 \leq p < \infty$. Then the following are equivalent:

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(ii) *B_w is chaotic;*

(iii) $\sum_n \frac{1}{\prod_{k=1}^n |w_k|^p} < \infty$.

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This is in stark contrast to the case of c_0. So what do the ℓ^p have that c_0 does not?

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A careful study of the proof of Bayart-Ruzsa leads to the following:

Theorem (Charpentier, Menet, GE, 2018...)

Let B_w be a weighted backward shift on a Banach sequence space in which the unit sequences $(e_n)_n$ form an unconditional basis. Suppose that this basis is boundedly complete. Then the following are equivalent:

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Bounded completeness means that the boundedness of $(\sum_{n=1}^N x_n e_n)_N$ in X implies the convergence of $\sum_{n=1}^\infty x_n e_n$.

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Strangely enough, the behaviour is again different for $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem (Charpentier, Menet, GE, 2018...)

On $H(\mathbb{D})$, weighted shifts are frequently hypercylic if and only if they are chaotic.

Question

Under which (general) properties on the space or the operator does frequent hypercyclicity imply chaos?

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Theorem (Menet 2017)

Let $X = \ell^p$, $1 \leq p < \infty$, or c_0. Then there is an operator T on X that is chaotic but not frequently hypercyclic.

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His operators are infinite matrices of the following form:

$$T = \begin{pmatrix}
0 & & & & * & & & & & & & \\
* & 0 & & & & & & & & & & \\
 & * & 0 & & * & & & & * & & & \\
 & & \ddots & 0 & & & & & & & & \\
 & & & * & 0 & & & & * & & & \\
 & & & & 0 & 0 & & & & & & \\
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Further recent advances

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Breaking news (Menet, 7 and 14 December 2018)

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As a consequence we have that

$$X = HC(T) + HC(T).$$

Hence $HC(T) \cup \{0\}$ can only be a linear subspace if $X = HC(T) \cup \{0\}$ (this can happen, but is rare...)

So, can $HC(T) \cup \{0\}$ contain a large linear subspace?

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YES

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YES - it's always the case!

Theorem (Herrero-Bourdon 1991/1993)

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Then $HC(T) \cup \{0\}$ contains a dense linear subspace.

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This question is by now quite well understood for general operators.

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Theorem (Falcó-GE 2018...)

(a) *Let B_w be a mixing weighted shift on ℓ^1 with the convolution product. Then $HC(B_w) \cup \{0\}$ contains a subalgebra that is not finitely generated.*

(b) *Let D be the differentiation operator on $H(\mathbb{C})$. Then $HC(D) \cup \{0\}$ contains a subalgebra that is not finitely generated.*

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The latter answers a question of Aron (2009) - also solved by Bès-Papathanasiou (2018...)

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