Role of Birkhoff-James orthogonality in the study of geometric properties of operator spaces

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* This is a joint work with Dr. D. Sain, Dr. P. Ghosh., A. Ray, and A. Mal

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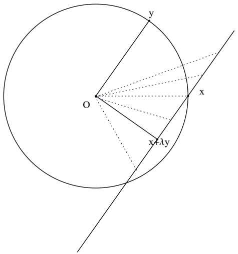

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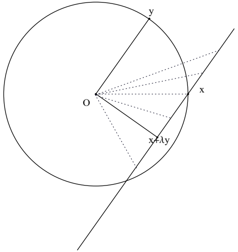

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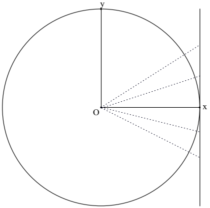

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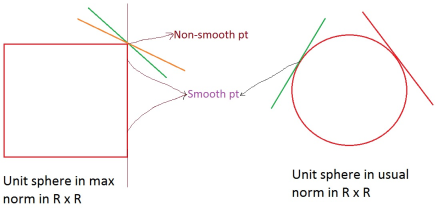
- The geometric interpretation of a smooth point in a two dimensional real normed linear space goes like this



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• So T is a smooth point iff $T \perp_B A_1$ and $T \perp_B A_2$ implies that $T \perp_B (A_1 + A_2)$.

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- **Remark.** In case $\mathbb{X}$ and $\mathbb{Y}$ are Hilbert spaces, T is smooth iff $M_T = \{\pm x_0\}$.

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Necessity: Let T be smooth and $M_T = \{\pm x_0\}$. If possible let Tx_0 is not smooth. Then there exists y, z such that $Tx_0 \perp_B y$, $Tx_0 \perp_B z$ but $Tx_0 \not\perp_B (y + z)$. Let $x_0 \perp_B H$. Define $A_1(\alpha x_0 + h) = \alpha y$ and $A_2(\alpha x_0 + h) = \alpha z$. Then $T \perp_B A_1$ and $T \perp_B A_2$ but $T \not\perp_B (A_1 + A_2)$, a contradiction.

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Theorem. Let $\mathbb{H}$ be a Hilbert space. Then $T \in B(\mathbb{H})$ is a smooth point iff $M_T = \{\pm x_0\}$ and $\|T\|_{x_0^\perp} < \|T\|$.

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Theorem. Let $\mathbb{X}, \mathbb{Y}$ be normed linear spaces. Let $T \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$ and $M_T = \{\pm x_0\}$. Further T satisfies the property : Given any $\delta > 0$, if $\{H_\alpha : \alpha \in \Lambda\}$ is the collection of all hyperspaces such that $d(x_0, H_\alpha) > \delta$ then $\sup\{\|Tx\| : x \in (\cup_\alpha H_\alpha) \cap S_{\mathbb{X}}\} < \|T\|$. Then for any $A \in \mathbb{B}(\mathbb{X}, \mathbb{Y})$, $T \perp_B A$ if and only if $Tx_0 \perp_B Ax_0$.

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In addition, if Tx_0 is smooth, then T is also smooth.

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This again gives a sufficient condition for the smoothness of an arbitrary bounded linear operator. For the necessary part we could obtain the following which is a little different from the sufficient part:

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(iii) $\sup\{\|Tx\| : x \in H_\alpha \cap S_{\mathbb{X}}\} < \|T\|$ for all $\alpha \in \Lambda$, where $\{H_{\alpha:\alpha\in\Lambda}\}$ is the collection of all hyperspaces such that $d(x_0, H_\alpha) > 0$.

Extreme points in the unit ball of a normed space

- An element $x \in S_{\mathbb{X}}$ is said to be an extreme point of $B_{\mathbb{X}}$ if for any $y, z \in B_{\mathbb{X}}$ and $t \in (0,1),\ x = (1-t)y + tz$ implies that $x = y = z$.

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The following theorem explores the role of norm attainment set of an operator on characterizing extreme contraction on a Banach space.

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The following theorem explores the role of norm attainment set of an operator on characterizing extreme contraction on a Banach space.

Theorem. Let $\mathbb{X}$ be a finite-dimensional polygonal Banach space and let $\mathbb{Y}$ be a strictly convex normed linear space. Then $T \in \mathbb{B}(X; Y)$ with $\|T\| = 1$ is an extreme contraction if and only if $\operatorname{span}(M_T \cap E_{\mathbb{X}}) = \mathbb{X}$.

References

1. Abatzoglou, T. J., *Norm derivatives on spaces of operators*, Math. Ann. **239** (1979), 129-135.

2. Bhatia, R. and Šemrl, P., *Orthogonality of matrices and distance problem*, Linear Alg. Appl. **287**,(1999) pp. 77-85.

3. Birkhoff, G., *Orthogonality in linear metric spaces*, Duke Math. J., **1**, (1935) pp. 169-172.

4. Deeb, W. and Khalil, R., *Exposed and smooth points of some classes of operators in* $L(l^p)$, Journal of Functional Analysis, **103** (2) (1992), 217-228.

5. Grzaslewicz, R. and Younis, R., *Smooth points and M-ideals*, Journal of Mathematical Analysis and Applications, **175** (1993) 91-95.

6. Ghosh P., Sain D. and Paul K., *orthogonality of bounded linear operators*, Linear Algebra and its Applications, **500** (2016), 43-51.

7. Heinrich S., *The differentiability of the norm in spaces of operators*, Functional. Anal, i Prilozen. (4) **9** (1975), 93-94. (English translation: Functional Anal. Appl. (4) 9 (1975), 360-362.)

8. Hennefeld J., *Smooth, compact operators*, Proc. Amer. Math. Soc., **77** (1) (1979), 87-90.

9. Holub, J.R., *On the metric geometry of ideals of operators on Hilbert space*, Math. Ann. **201** (1973), 157-163.

10. James C. R., *Orthogonality and linear functionals in normed linear spaces*, Transactions of the American Mathematical Society, Vol. **61** (1947) 265-292.

11. James C. R., *Inner products in normed linear spaces*, Bull. Amer. Math. Soc., **53** (1947a), 559-566.

12. Kittaneh, F. and Younis, R., *Smooth points of certain operator spaces*, Int. Equations and Operator Theory **13** (1990), 849-855.

13. Li, C.K. and Schneider, H., *Orthogonality of matrices*, Linear Alg. Appl., **47**,(2002) pp. 115-122.

14. Paul, K., *Translatable radii of an operator in the direction of another operator*, Scientiae Mathematicae **2** (1999), 119-122.

15. Paul, K., Sain, D. and Ghosh, P., *Birkhoff-James orthogonality and smoothness of bounded linear operators*, Linear Algebra and its Applications, **506** (2016) 551-563.

16. Paul, K., Sain, D. and Jha K., *On strong orthogonality and strictly convex normed linear space*, Journal of Inequalities and Application, (2013) 2013:242.

17. Sain, D. and Paul, K., *Operator norm attainment and inner product spaces*, Linear Algebra and its Applications, **439** (2013) 2448–2452.

18. Sain, D., Paul, K. and Mal, A., *A complete characterization of Birkhoff-James orthogonality in infinite*

dimensional normed space , Journal of Operator Theory, **80** (2018) 399-413.

19. Sain, D., Paul, K. and Hait, S., *Operator norm attainment and Birkhoff-James orthogonality*, Linear Algebra and its Applications, **476** (2015) 85–97.

20. Sain, D., Ray, A. and Paul, K., *Extreme contractions on finite-dimensional polygonal Banach spaces*, Journal of Convex Analysis, **26** no. 3 (2019) [Final page numbers is not yet available].

21. Werner, W., *Smooth points in some spaces of bounded operators*, Integral Equation Operator Theory, **15** (1992) 496-502.

Thank You