"On distribution of sums of weakly monotone position operators."

Janusz Wysoczański
Institute of Mathematics, Wrocław University
in collaboration with
Vitonofrio Crismale and Maria Elena Griseta
(University of Bari)

To appear in Journal of Theoretical Probability

OTOA, ISI Bangalore, December 13–19, 2018

Plan

1. Weakly Monotone Creation and Annihilation Operators
2. Partial isometries and commutation relations
3. Monotone Independence
4. Distribution of sum of position operators
5. Combinatorics of moments
6. Weakly monotone ordered partitions

Construction - JW (2005)

Weakly Monotone Fock Space

Consider Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_n : n \in \mathbb{N}\} \subset \mathcal{H}$.

Construction - JW (2005)

Weakly Monotone Fock Space

Consider Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_n : n \in \mathbb{N}\} \subset \mathcal{H}$. By $\mathcal{F}_{wm}(\mathcal{H})$ we denote the subspace of the full Fock space on $\mathcal{H}$ spanned by the vacuum vector Ω and the simple tensors of the form

$$e_{i_n} \otimes e_{i_{n-1}} \otimes \cdots \otimes e_{i_1}, \quad \text{where} \quad i_n \geq i_{n-1} \geq \cdots \geq i_1.$$

Construction - JW (2005)

Weakly Monotone Fock Space

Consider Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_n : n \in \mathbb{N}\} \subset \mathcal{H}$. By $\mathcal{F}_{wm}(\mathcal{H})$ we denote the subspace of the full Fock space on $\mathcal{H}$ spanned by the vacuum vector Ω and the simple tensors of the form

$$e_{i_n} \otimes e_{i_{n-1}} \otimes \cdots \otimes e_{i_1}, \quad \text{where} \quad i_n \geq i_{n-1} \geq \cdots \geq i_1 .$$

scalar product:

$$\langle e_{k_r} \otimes \cdots \otimes e_{k_1}, e_{j_s} \otimes \cdots \otimes e_{j_1} \rangle = \begin{cases} 1 & \text{if} \quad r{=}s, k_1 = j_1, \ldots, k_r = j_r \\ 0 & \text{otherwise} \end{cases}$$

Construction - JW (2005)

Weakly Monotone Fock Space

Consider Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_n : n \in \mathbb{N}\} \subset \mathcal{H}$. By $\mathcal{F}_{wm}(\mathcal{H})$ we denote the subspace of the full Fock space on $\mathcal{H}$ spanned by the vacuum vector Ω and the simple tensors of the form

$$e_{i_n} \otimes e_{i_{n-1}} \otimes \cdots \otimes e_{i_1}, \quad \text{where} \quad i_n \geq i_{n-1} \geq \cdots \geq i_1.$$

scalar product:

$$\langle e_{k_r} \otimes \cdots \otimes e_{k_1}, e_{j_s} \otimes \cdots \otimes e_{j_1} \rangle = \begin{cases} 1 & \text{if} \quad r{=}s, k_1 = j_1, \ldots, k_r = j_r \\ 0 & \text{otherwise} \end{cases}$$

Remark

The indexes are weakly decreasing, hence *weakly monotone* Fock space.

Creation and Annihilation Operators

Creation

Creation $A_j^\dagger := A^\dagger(e_j)$ by the basis vector e_j is defined by

$$A_j^\dagger \Omega \quad = \quad e_j,$$

Creation and Annihilation Operators

Creation

Creation $A_j^\dagger := A^\dagger(e_j)$ by the basis vector e_j is defined by

$$
\begin{aligned}
A_j^\dagger \Omega &= e_j, \\
A_j^\dagger (e_{i_n} \otimes e_{i_{n-1}} \otimes \cdots \otimes e_{i_1}) &= e_j \otimes e_{i_n} \otimes e_{i_{n-1}} \otimes \cdots \otimes e_{i_1} \\
\text{if} \quad j &\geq i_n
\end{aligned}
$$

Creation and Annihilation Operators

Creation

Creation $A_j^\dagger := A^\dagger(e_j)$ by the basis vector e_j is defined by

$$
\begin{aligned}
A_j^\dagger \Omega &= e_j, \\
A_j^\dagger(e_{i_n} \otimes e_{i_{n-1}} \otimes \cdots \otimes e_{i_1}) &= e_j \otimes e_{i_n} \otimes e_{i_{n-1}} \otimes \cdots \otimes e_{i_1} \\
\text{if} \quad j &\geq i_n \\
A_j^\dagger(e_{i_n} \otimes e_{i_{n-1}} \otimes \cdots \otimes e_{i_1}) &= 0 \quad \text{if} \quad j < i_n.
\end{aligned}
$$

Creation and Annihilation Operators

Creation

Creation $A_j^\dagger := A^\dagger(e_j)$ by the basis vector e_j is defined by

$$\begin{aligned} A_j^\dagger \Omega &= e_j, \\ A_j^\dagger(e_{i_n} \otimes e_{i_{n-1}} \otimes \cdots \otimes e_{i_1}) &= e_j \otimes e_{i_n} \otimes e_{i_{n-1}} \otimes \cdots \otimes e_{i_1} \\ \text{if} \quad j &\geq i_n \\ A_j^\dagger(e_{i_n} \otimes e_{i_{n-1}} \otimes \cdots \otimes e_{i_1}) &= 0 \quad \text{if} \quad j < i_n. \end{aligned}$$

Annihilation

Annihilation A_j by the basis vector e_j is defined by

$$A_j(e_i) = \delta_{ji}\Omega, \quad A_j(e_{i_n} \otimes e_{i_{n-1}} \otimes \cdots \otimes e_{i_1}) = \delta_{ji_n} e_{i_{n-1}} \otimes \cdots \otimes e_{i_1}$$

Partial isometries

The operators $A_j A_j^\dagger$ and $A_j^\dagger A_j$ are orthogonal projections, hence the creation operators $A_j^\dagger$ and the annilhilation operators A_j are partial isometries

Partial isometries

The operators $A_j A_j^\dagger$ and $A_j^\dagger A_j$ are orthogonal projections, hence the creation operators $A_j^\dagger$ and the annilhilation operators A_j are partial isometries

Orthogonal projections $A_j^\dagger A_j$

map $\mathcal{F}_{wm}(\mathcal{H})$ onto the subspace $\mathcal{F}_{wm}(\mathcal{H})_{=j}$ spanned by the vacuum and simple tensors of the form (starting with e_j)

$$e_j \otimes e_{i_k} \otimes \cdots \otimes e_{i_1} \quad \text{where} \quad j \geq i_k \geq \ldots \geq i_1$$

Partial isometries

The operators $A_j A_j^\dagger$ and $A_j^\dagger A_j$ are orthogonal projections, hence the creation operators $A_j^\dagger$ and the annilhilation operators A_j are partial isometries

Orthogonal projections $A_j^\dagger A_j$

map $\mathcal{F}_{wm}(\mathcal{H})$ onto the subspace $\mathcal{F}_{wm}(\mathcal{H})_{=j}$ spanned by the vacuum and simple tensors of the form (starting with e_j)

$$e_j \otimes e_{i_k} \otimes \cdots \otimes e_{i_1} \quad \text{where} \quad j \geq i_k \geq \ldots \geq i_1$$

Orthogonal projections $A_j A_j^\dagger$

map $\mathcal{F}_{wm}(\mathcal{H})$ onto the subspace $\mathcal{F}_{wm}(\mathcal{H})_{\leq j}$ spanned by the vacuum and simple tensors of the form (starting with e_{i_k} for any $i_k \leq j$)

$$e_{i_k} \otimes \cdots \otimes e_{i_1} \quad \text{with} \quad j \geq i_k \geq \ldots \geq i_1$$

Relations

Creation and annihilation operators are bounded and mutually adjoint: $(A_j)^* = A_j^\dagger$ and satisfy relations:

Relations

Creation and annihilation operators are bounded and mutually adjoint: $(A_j)^* = A_j^\dagger$ and satisfy relations:

Relations

$$A_k A_j = 0 \qquad \text{if} \quad j < k$$

Relations

Creation and annihilation operators are bounded and mutually adjoint: $(A_j)^* = A_j^\dagger$ and satisfy relations:

Relations

$$\begin{aligned} A_k A_j &= 0 \qquad \text{if} \quad j < k \\ A_k^\dagger A_j^\dagger &= 0 \qquad \text{if} \quad j > k \end{aligned}$$

Relations

Creation and annihilation operators are bounded and mutually adjoint: $(A_j)^* = A_j^\dagger$ and satisfy relations:

Relations

$$\begin{aligned} A_k A_j &= 0 \qquad \text{if} \quad j < k \\ A_k^\dagger A_j^\dagger &= 0 \qquad \text{if} \quad j > k \\ A_j A_k^\dagger &= 0 \qquad \text{if} \quad k \neq j \end{aligned}$$

Relations

Creation and annihilation operators are bounded and mutually adjoint: $(A_j)^* = A_j^\dagger$ and satisfy relations:

Relations

$$
\begin{aligned}
A_k A_j &= 0 \qquad \text{if} \quad j < k \\
A_k^\dagger A_j^\dagger &= 0 \qquad \text{if} \quad j > k \\
A_j A_k^\dagger &= 0 \qquad \text{if} \quad k \neq j
\end{aligned}
$$

More relations ($j, m \in \mathbb{N}$)

$$
A_j A_j^\dagger \;=\; (A_j)^m (A_j^\dagger)^m,
$$

Commutation relations

Commutation relations

$$I - A_j A_j^\dagger = \sum_{k>j} A_k^\dagger A_k, \quad n = \dim(\mathcal{H}) = \infty$$

Commutation relations

Commutation relations

$$
\begin{aligned}
I - A_j A_j^\dagger &= \sum_{k>j} A_k^\dagger A_k, \quad n = \dim(\mathcal{H}) = \infty \\
I_n - P_\Omega &= \sum_{k=1}^{n} A_k^\dagger A_k, \quad n = \dim(\mathcal{H}) < \infty
\end{aligned}
$$

Commutation relations

Commutation relations

$$
\begin{aligned}
I - A_j A_j^\dagger &= \sum_{k>j} A_k^\dagger A_k, \quad n = \dim(\mathcal{H}) = \infty \\
I_n - P_\Omega &= \sum_{k=1}^{n} A_k^\dagger A_k, \quad n = \dim(\mathcal{H}) < \infty
\end{aligned}
$$

Problem

Describe $C^*(\{A_j^\dagger, A_j : j \leq n = \dim(\mathcal{H}) \leq \infty\})$, i.e. the C^*-algebra generated by the weakly monotone creation and annihilation operators (i.e. by partial isometries).

Monotone independence of algebras

Monotone independence - N. MURAKI 2001

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, i.e. $\mathcal{A}$ is a unital *-algebra and φ is a state on $\mathcal{A}$. We say that a family $\{\mathcal{A}_j : j \in \mathbb{N}\}$ of subalgebras of $\mathcal{A}$ is monotone independent in $(\mathcal{A}, \varphi)$ if the following two condition hold:

Monotone independence of algebras

Monotone independence - N. MURAKI 2001

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, i.e. $\mathcal{A}$ is a unital *-algebra and φ is a state on $\mathcal{A}$. We say that a family $\{\mathcal{A}_j : j \in \mathbb{N}\}$ of subalgebras of $\mathcal{A}$ is monotone independent in $(\mathcal{A}, \varphi)$ if the following two condition hold:

[M1]

If $i < j > k$ and $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$, $c \in \mathcal{A}_k$ then $abc = \varphi(b)ac$ (local index maxima get out)

Monotone independence of algebras

Monotone independence - N. MURAKI 2001

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, i.e. $\mathcal{A}$ is a unital *-algebra and φ is a state on $\mathcal{A}$. We say that a family $\{\mathcal{A}_j : j \in \mathbb{N}\}$ of subalgebras of $\mathcal{A}$ is monotone independent in $(\mathcal{A}, \varphi)$ if the following two condition hold:

[M1]

If $i < j > k$ and $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$, $c \in \mathcal{A}_k$ then $abc = \varphi(b)ac$ (local index maxima get out)

[M2]

If $j_1 > \ldots > j_k < \ldots < j_n$ and $a_i \in \mathcal{A}_{j_i}$ then

$$\varphi(a_1 a_2 \ldots a_n) = \prod_{i=1}^{n} \varphi(a_i)$$

Monotone independence of position operators

Let $\mathcal{B}$ be the unital $*$-algebra of all bounded operators on the weakly monotone Fock space $\mathcal{F}_{wm}(\mathcal{H})$ and let φ be the vacuum state: $\varphi(b) := \langle b\Omega, \Omega\rangle$ for $b \in \mathcal{B}$.

Monotone independence of position operators

Let $\mathcal{B}$ be the unital $*$-algebra of all bounded operators on the weakly monotone Fock space $\mathcal{F}_{wm}(\mathcal{H})$ and let φ be the vacuum state: $\varphi(b) := \langle b\Omega, \Omega\rangle$ for $b \in \mathcal{B}$. For each $j \in \mathbb{N}$ consider the $*$-subalgebra $\mathcal{B}_j := \mathrm{alg}\{A_j,\ A_j^\dagger\} \subset \mathcal{B}$ generated by the creation $A_j^\dagger$ and annihilation A_j operators.

Monotone independence of position operators

Let $\mathcal{B}$ be the unital $*$-algebra of all bounded operators on the weakly monotone Fock space $\mathcal{F}_{wm}(\mathcal{H})$ and let φ be the vacuum state: $\varphi(b) := \langle b\Omega, \Omega\rangle$ for $b \in \mathcal{B}$. For each $j \in \mathbb{N}$ consider the $*$-subalgebra $\mathcal{B}_j := \mathrm{alg}\{A_j,\ A_j^\dagger\} \subset \mathcal{B}$ generated by the creation $A_j^\dagger$ and annihilation A_j operators.

Monotone independence

The algebras $\{\mathcal{B}_j\ : j \in \mathbb{N}\}$ are monotone independent in the non-commutative probability space $(\mathcal{B}, \varphi)$.

Monotone independence of position operators

Let $\mathcal{B}$ be the unital $*$-algebra of all bounded operators on the weakly monotone Fock space $\mathcal{F}_{wm}(\mathcal{H})$ and let φ be the vacuum state: $\varphi(b) := \langle b\Omega, \Omega\rangle$ for $b \in \mathcal{B}$. For each $j \in \mathbb{N}$ consider the $*$-subalgebra $\mathcal{B}_j := \mathrm{alg}\{A_j,\ A_j^{\dagger}\} \subset \mathcal{B}$ generated by the creation $A_j^{\dagger}$ and annihilation A_j operators.

Monotone independence

The algebras $\{\mathcal{B}_j\ : j \in \mathbb{N}\}$ are monotone independent in the non-commutative probability space $(\mathcal{B}, \varphi)$.

Corollary

For $j \in \mathbb{N}$ the position operators $G_j := A_j + A_j^{\dagger}$ are monotone independent in $(\mathcal{B}, \varphi)$.

Main object of our study

Sums of position operators

We study the distributions

$$\varphi((S_p)^m), \quad m \in \mathbb{N}$$

of the sums

$$S_p := G_1 + G_2 + \cdots + G_p = \sum_{j=1}^{p} \left(A_j + A_j^{\dagger} \right)$$

of position operators, for $p \in \mathbb{N}$.

Main object of our study

Sums of position operators

We study the distributions

$$\varphi((S_p)^m), \quad m \in \mathbb{N}$$

of the sums

$$S_p := G_1 + G_2 + \cdots + G_p = \sum_{j=1}^{p} \left(A_j + A_j^\dagger \right)$$

of position operators, for $p \in \mathbb{N}$.

Since all G_j's have the same distribution, hence the law of S_p will be the p-fold monotone convolution of the single distribution of G_1.

Distribution of sums of position operators

Distribution of a single position operator

The distribution of each position operator $G_j := A_j + A_j^\dagger$ is the Wigner semicircle law $W(x) := \frac{\sqrt{4-x^2}}{2\pi}$ on $[-2,\ 2]$, since

$$\varphi((G_j)^{2m}) = C_m = \frac{1}{m+1}\binom{2m}{m}, \quad \varphi((G_j)^{2m+1}) = 0, \quad m \geq 0.$$

Distribution of sums of position operators

Distribution of a single position operator

The distribution of each position operator $G_j := A_j + A_j^\dagger$ is the Wigner semicircle law $W(x) := \frac{\sqrt{4-x^2}}{2\pi}$ on $[-2,\ 2]$, since

$$\varphi((G_j)^{2m}) = C_m = \frac{1}{m+1}\binom{2m}{m}, \quad \varphi((G_j)^{2m+1}) = 0, \quad m \geq 0.$$

Monotone convolution of semicirlce law

The distribution of the sum $S_p := \sum_{j=1}^{p} G_j$ of p position operators is the p-th monotone convolution power of the semicircle law:

$$W_p := \underbrace{W \triangleright \cdots \triangleright W}_{p}$$

Distribution of sum of two position operators

Theorem: distribution of $S_2 = G_1 + G_2$

The distribution of S_2 is absolutely continuous with density

$$W_2(x) = \begin{cases} \frac{1}{4\pi}\left(\sqrt{\sqrt{100-16x^2}-x^2+10}-\sqrt{4-x^2}\right) & |x| \leq 2 \\ & \\ & \end{cases}$$

Distribution of sum of two position operators

Theorem: distribution of $S_2 = G_1 + G_2$

The distribution of S_2 is absolutely continuous with density

$$W_2(x) = \begin{cases} \frac{1}{4\pi}\left(\sqrt{\sqrt{100-16x^2}-x^2+10}-\sqrt{4-x^2}\right) & |x| \leq 2 \\ \frac{1}{4\pi}\sqrt{-2x^2-2|x|\sqrt{x^2-4}+20}, & 2 < |x| \leq \frac{5}{2} \end{cases}$$

Distribution of sum of two position operators

Theorem: distribution of $S_2 = G_1 + G_2$

The distribution of S_2 is absolutely continuous with density

$$W_2(x) = \begin{cases} \frac{1}{4\pi}\left(\sqrt{\sqrt{100-16x^2}-x^2+10}-\sqrt{4-x^2}\right) & |x| \le 2 \\ \frac{1}{4\pi}\sqrt{-2x^2-2|x|\sqrt{x^2-4}+20}, & 2 < |x| \le \frac{5}{2} \\ 0 & |x| \ge \frac{5}{2} \end{cases}$$

Distribution of sum of two position operators

Theorem: distribution of $S_2 = G_1 + G_2$

The distribution of S_2 is absolutely continuous with density

$$W_2(x) = \begin{cases} \frac{1}{4\pi}\left(\sqrt{\sqrt{100-16x^2}-x^2+10}-\sqrt{4-x^2}\right) & |x| \leq 2 \\ \frac{1}{4\pi}\sqrt{-2x^2-2|x|\sqrt{x^2-4}+20}, & 2 < |x| \leq \frac{5}{2} \\ 0 & |x| \geq \frac{5}{2} \end{cases}$$

In particular, at the distinguished points $W_2(\pm 2) = 2\sqrt{3}$ and $W_2(\pm\frac{5}{2}) = 0$.

Distribution of sum of p position operators

Theorem: distribution W_p of $S_p := G_1 + G_2 + \cdots + G_p$

The distribution of S_p is an absolutely continuous probability measure with density $W_p := W^{\triangleright p}$.

Distribution of sum of p position operators

Theorem: distribution W_p of $S_p := G_1 + G_2 + \cdots + G_p$

The distribution of S_p is an absolutely continuous probability measure with density $W_p := W^{\triangleright p}$. The Cauchy transform of W_p defined by

$$\mathcal{G}_p(z) := \int_{-\infty}^{+\infty} \frac{W_p(x)\ dx}{z - x},$$

satisfies the recursion

$$\mathcal{G}_p^2(z) + \mathcal{G}_p(z) \cdot (K_{p-1}(z) - z) + 1 \quad = \quad 0,$$

Distribution of sum of p position operators

Theorem: distribution W_p of $S_p := G_1 + G_2 + \cdots + G_p$

The distribution of S_p is an absolutely continuous probability measure with density $W_p := W^{\triangleright p}$. The Cauchy transform of W_p defined by

$$\mathcal{G}_p(z) := \int_{-\infty}^{+\infty} \frac{W_p(x)\ dx}{z - x},$$

satisfies the recursion

$$\begin{aligned} \mathcal{G}_p^2(z) + \mathcal{G}_p(z) \cdot (K_{p-1}(z) - z) + 1 &= 0, \\ \text{where} \quad \sum_{k=1}^{p-1} \mathcal{G}_k(z) &=: K_{p-1}(z). \end{aligned}$$

Distribution of sum of p position operators

Theorem: support of the distribution W_p

The support of the measure W_p is a symmetric interval $[-a_p,\ a_p]$ where the right-end points satisfy the recursion

$$a_{p+1} = a_p + \frac{1}{a_p}.$$

Distribution of sum of p position operators

Theorem: support of the distribution W_p

The support of the measure W_p is a symmetric interval $[-a_p,\ a_p]$ where the right-end points satisfy the recursion

$$a_{p+1} = a_p + \frac{1}{a_p}.$$

Tools for proof

Consider *Zhukovsky map* $Z(w) := w + \frac{1}{w}$, its p-fold compositions Z_p and the *reciprocal Cauchy transforms* $\mathcal{F}_p(z) := \frac{1}{\mathcal{G}_p(z)}$ of W_p.

Distribution of sum of p position operators

Theorem: support of the distribution W_p

The support of the measure W_p is a symmetric interval $[-a_p,\ a_p]$ where the right-end points satisfy the recursion

$$a_{p+1} = a_p + \frac{1}{a_p}.$$

Tools for proof

Consider *Zhukovsky map* $Z(w) := w + \frac{1}{w}$, its p-fold compositions Z_p and the *reciprocal Cauchy transforms* $\mathcal{F}_p(z) := \frac{1}{\mathcal{G}_p(z)}$ of W_p. Then we have

$$Z_p(\mathcal{F}_p(z)) = z, \quad \text{and}$$

Distribution of sum of p position operators

Theorem: support of the distribution W_p

The support of the measure W_p is a symmetric interval $[-a_p, a_p]$ where the right-end points satisfy the recursion

$$a_{p+1} = a_p + \frac{1}{a_p}.$$

Tools for proof

Consider *Zhukovsky map* $Z(w) := w + \frac{1}{w}$, its p-fold compositions Z_p and the *reciprocal Cauchy transforms* $\mathcal{F}_p(z) := \frac{1}{\mathcal{G}_p(z)}$ of W_p. Then we have

$$Z_p(\mathcal{F}_p(z)) = z, \quad \text{and} \quad \mathcal{F}_p(a_p) = 1,$$

Distribution of sum of p position operators

Theorem: support of the distribution W_p

The support of the measure W_p is a symmetric interval $[-a_p,\ a_p]$ where the right-end points satisfy the recursion

$$a_{p+1} = a_p + \frac{1}{a_p}.$$

Tools for proof

Consider *Zhukovsky map* $Z(w) := w + \frac{1}{w}$, its p-fold compositions Z_p and the *reciprocal Cauchy transforms* $\mathcal{F}_p(z) := \frac{1}{\mathcal{G}_p(z)}$ of W_p. Then we have

$$Z_p(\mathcal{F}_p(z)) = z, \quad \text{and} \quad \mathcal{F}_p(a_p) = 1, \quad \mathcal{F}_p(-a_p) = -1.$$

Distribution of sum of p position operators

Estimate for the support of W_p

In particular, we have the estimate ($\mathrm{supp}(W_p) = [-a_p, \ a_p]$)

$$\sqrt{p + \sqrt{p(p+1)}} \leq a_p \leq \sqrt{2p + \sqrt{2p}}$$

Distribution of sum of p position operators

Estimate for the support of W_p

In particular, we have the estimate ($\mathrm{supp}(W_p) = [-a_p, a_p]$)

$$\sqrt{p + \sqrt{p(p+1)}} \leq a_p \leq \sqrt{2p + \sqrt{2p}}$$

hence

$$\frac{a_p}{\sqrt{p}} \geq \sqrt{2} \quad \text{and} \qquad \lim_{p\to\infty} \frac{a_p}{\sqrt{p}} = \sqrt{2}.$$

Distribution of sum of p position operators

Estimate for the support of W_p

In particular, we have the estimate $(\mathrm{supp}(W_p) = [-a_p,\ a_p])$

$$\sqrt{p + \sqrt{p(p+1)}} \leq a_p \leq \sqrt{2p + \sqrt{2p}}$$

hence

$$\frac{a_p}{\sqrt{p}} \geq \sqrt{2} \quad \text{and} \qquad \lim_{p\to\infty} \frac{a_p}{\sqrt{p}} = \sqrt{2}.$$

Therefore the scaled supports $\left[-\frac{a_p}{\sqrt{p}},\ \frac{a_p}{\sqrt{p}}\right]$ form an ascending sequence of intervals with intersection $[-\sqrt{2},\ \sqrt{2}]$:

Distribution of sum of p position operators

Estimate for the support of W_p

In particular, we have the estimate $(\mathrm{supp}(W_p) = [-a_p,\ a_p])$

$$\sqrt{p+\sqrt{p(p+1)}} \leq a_p \leq \sqrt{2p+\sqrt{2p}}$$

hence

$$\frac{a_p}{\sqrt{p}} \geq \sqrt{2} \quad \text{and} \quad \lim_{p\to\infty} \frac{a_p}{\sqrt{p}} = \sqrt{2}.$$

Therefore the scaled supports $\left[-\frac{a_p}{\sqrt{p}},\ \frac{a_p}{\sqrt{p}}\right]$ form an ascending sequence of intervals with intersection $[-\sqrt{2},\ \sqrt{2}]$:

$$\frac{a_{p+1}}{\sqrt{p+1}} < \frac{a_p}{\sqrt{p}}$$

Distribution of sum of p position operators

Estimate for the support of W_p

In particular, we have the estimate ($\mathrm{supp}(W_p) = [-a_p,\ a_p]$)

$$\sqrt{p+\sqrt{p(p+1)}} \le a_p \le \sqrt{2p+\sqrt{2p}}$$

hence

$$\frac{a_p}{\sqrt{p}} \ge \sqrt{2} \quad \text{and} \quad \lim_{p\to\infty} \frac{a_p}{\sqrt{p}} = \sqrt{2}.$$

Therefore the scaled supports $\left[-\frac{a_p}{\sqrt{p}},\ \frac{a_p}{\sqrt{p}}\right]$ form an ascending sequence of intervals with intersection $[-\sqrt{2},\ \sqrt{2}]$:

$$\frac{a_{p+1}}{\sqrt{p+1}} < \frac{a_p}{\sqrt{p}} \iff \sqrt{p+\sqrt{p(p+1)}} \le a_p.$$

Moments of $S_2 = G_1 + G_2$

Since the odd moments vanish, we consider only the even ones.

Moments of $S_2 = G_1 + G_2$

Since the odd moments vanish, we consider only the even ones.

Recursion for $p = 2$

The (even) moment sequence $d_n^{(2)} := \varphi((S_2)^{2n})$ satisfies $d_0^{(2)} = 1$ and

$$d_n^{(2)} = \sum_{k=1}^{n} d_{n-k}^{(2)} \left(d_{k-1}^{(2)} + C_{k-1} \right),$$

Moments of $S_2 = G_1 + G_2$

Since the odd moments vanish, we consider only the even ones.

Recursion for $p = 2$

The (even) moment sequence $d_n^{(2)} := \varphi((S_2)^{2n})$ satisfies $d_0^{(2)} = 1$ and

$$d_n^{(2)} = \sum_{k=1}^{n} d_{n-k}^{(2)} \left(d_{k-1}^{(2)} + C_{k-1} \right),$$

$$C_k := \frac{1}{k+1}\binom{2k}{k} \text{ Catalan numbers.}$$

Moments of $S_p = G_1 + G_2 + \cdots + G_p$

Theorem: recursion for arbitrary $p \in \mathbb{N}$

The (even) moment sequences $d_n^{(p)} := \varphi((S_p)^{2n})$, defined for $p, n \in \mathbb{N}$ ($p \geq 1$, $n \geq 0$), satisfy the recursion

$$d_0^{(p)} = 1, \quad (n = 0),$$

Moments of $S_p = G_1 + G_2 + \cdots + G_p$

Theorem: recursion for arbitrary $p \in \mathbb{N}$

The (even) moment sequences $d_n^{(p)} := \varphi((S_p)^{2n})$, defined for $p, n \in \mathbb{N}$ ($p \geq 1$, $n \geq 0$), satisfy the recursion

$$\begin{aligned} d_0^{(p)} &= 1, \quad (n = 0), \\ d_n^{(1)} &= C_n, \quad (p = 1), \end{aligned}$$

Moments of $S_p = G_1 + G_2 + \cdots + G_p$

Theorem: recursion for arbitrary $p \in \mathbb{N}$

The (even) moment sequences $d_n^{(p)} := \varphi((S_p)^{2n})$, defined for $p, n \in \mathbb{N}$ ($p \geq 1$, $n \geq 0$), satisfy the recursion

$$\begin{aligned} d_0^{(p)} &= 1, \quad (n = 0), \\ d_n^{(1)} &= C_n, \quad (p = 1), \\ d_n^{(p)} &= \sum_{k=1}^{n} d_{n-k}^{(p)} \cdot \left(\sum_{j=1}^{p} d_{k-1}^{(j)} \right). \end{aligned}$$

Moments of $S_p = G_1 + G_2 + \cdots + G_p$

Theorem: recursion for arbitrary $p \in \mathbb{N}$

The (even) moment sequences ${d_n}^{(p)} := \varphi((S_p)^{2n})$, defined for $p, n \in \mathbb{N}$ ($p \geq 1$, $n \geq 0$), satisfy the recursion

$$\begin{aligned} d_0^{(p)} &= 1, \quad (n = 0), \\ d_n^{(1)} &= C_n, \quad (p = 1), \\ {d_n}^{(p)} &= \sum_{k=1}^{n} d_{n-k}^{(p)} \cdot \left(\sum_{j=1}^{p} d_{k-1}^{(j)} \right). \end{aligned}$$

Moments of $S_p = G_1 + G_2 + \cdots + G_p$

Examples - Franz Lehner's computer calculations

Here are some examples of the moment sequences $d_n^{(p)} := \varphi((S_p)^{2n})$ for $p = 1, 2, \ldots,\ n = 0, 1, 2, \ldots$:

$d_n^{(1)} = 1, \quad 1, \quad 2, \quad 5, \quad 14, \quad 42, \quad 132,$ (*Catalan*)

Moments of $S_p = G_1 + G_2 + \cdots + G_p$

Examples - Franz Lehner's computer calculations

Here are some examples of the moment sequences $d_n^{(p)} := \varphi((S_p)^{2n})$ for $p = 1, 2, \ldots,\ n = 0, 1, 2, \ldots$:

$$\begin{array}{lcrrrrrrrl} d_n^{(1)} & = & 1, & 1, & 2, & 5, & 14, & 42, & 132, & (\textit{Catalan}) \\ d_n^{(2)} & = & 1, & 2, & 7, & 29, & 131, & 625, & 3099, & (A007852) \end{array}$$

Moments of $S_p = G_1 + G_2 + \cdots + G_p$

Examples - Franz Lehner's computer calculations

Here are some examples of the moment sequences $d_n^{(p)} := \varphi((S_p)^{2n})$ for $p = 1, 2, \ldots,\ n = 0, 1, 2, \ldots$:

$$
\begin{array}{llrrrrrrrl}
d_n^{(1)} & = & 1, & 1, & 2, & 5, & 14, & 42, & 132, & (\mathit{Catalan}) \\
d_n^{(2)} & = & 1, & 2, & 7, & 29, & 131, & 625, & 3099, & (A007852) \\
d_n^{(3)} & = & 1, & 3, & 15, & 87, & 544, & 3566, & 24165, & \ldots \\
d_n^{(4)} & = & 1, & 4, & 26, & 194, & 1551, & 12944, & 111313, & \ldots \\
d_n^{(5)} & = & 1, & 5, & 40, & 365, & 3555, & 36045, & 375797, & \ldots \\
d_n^{(6)} & = & 1, & 6, & 57, & 615, & 7064, & 84307, & 1033089, & \ldots \\
d_n^{(7)} & = & 1, & 7, & 77, & 959, & 12691, & 174265, & 2454221, & \ldots
\end{array}
$$

Moments of $S_p = G_1 + G_2 + \cdots + G_p$

Examples - Franz Lehner's computer calculations

Here are some examples of the moment sequences $d_n^{(p)} := \varphi((S_p)^{2n})$ for $p = 1, 2, \ldots,\ n = 0, 1, 2, \ldots$:

$$
\begin{array}{rcrrrrrrrl}
d_n^{(1)} & = & 1, & 1, & 2, & 5, & 14, & 42, & 132, & (\mathit{Catalan}) \\
d_n^{(2)} & = & 1, & 2, & 7, & 29, & 131, & 625, & 3099, & (A007852) \\
d_n^{(3)} & = & 1, & 3, & 15, & 87, & 544, & 3566, & 24165, & \ldots \\
d_n^{(4)} & = & 1, & 4, & 26, & 194, & 1551, & 12944, & 111313, & \ldots \\
d_n^{(5)} & = & 1, & 5, & 40, & 365, & 3555, & 36045, & 375797, & \ldots \\
d_n^{(6)} & = & 1, & 6, & 57, & 615, & 7064, & 84307, & 1033089, & \ldots \\
d_n^{(7)} & = & 1, & 7, & 77, & 959, & 12691, & 174265, & 2454221, & \ldots
\end{array}
$$

$d_n^{(2)}$: A007852 (Sloan) *Antichains (totally disordered subsets) in rooted plane trees on n nodes.*

Properties of the moment sequences

Theorem: polynomials

For fixed $n \in \mathbb{N}$ the numbers $\left(d_n^{(p)}\right)_{p\geq 1}$ are polynomials in the variable p of degree n.

Properties of the moment sequences

Theorem: polynomials

For fixed $n \in \mathbb{N}$ the numbers $\left(d_n^{(p)}\right)_{p\geq 1}$ are polynomials in the variable p of degree n.

Remark

Proof by using Hasebe's monotone cumulants of Wigner law

Properties of the moment sequences

Theorem: polynomials

For fixed $n \in \mathbb{N}$ the numbers $\left(d_n^{(p)}\right)_{p \geq 1}$ are polynomials in the variable p of degree n.

Remark

Proof by using Hasebe's monotone cumulants of Wigner law or by the Faulhaber formula (r-h.s. is polynomial of degree n in p)

$$\sum_{j=1}^{p} j^{n-1} = \frac{1}{n}\sum_{i=0}^{n-1}(-1)^i \binom{n}{i} B_i p^{n-i} = \frac{p^n}{n} + \dots,$$

Properties of the moment sequences

Theorem: polynomials

For fixed $n \in \mathbb{N}$ the numbers $\left(d_n^{(p)}\right)_{p\geq 1}$ are polynomials in the variable p of degree n.

Remark

Proof by using Hasebe's monotone cumulants of Wigner law or by the Faulhaber formula (r-h.s. is polynomial of degree n in p)

$$\sum_{j=1}^{p} j^{n-1} = \frac{1}{n}\sum_{i=0}^{n-1}(-1)^i \binom{n}{i} B_i p^{n-i} = \frac{p^n}{n} + \dots,$$

$$\frac{z}{e^z - 1} = \sum_{j=0}^{\infty} B_j \frac{z^j}{j!}, \quad B_j - \text{Bernoulli numbers}$$

Properties of the moment sequences

Examples - Franz Lehner's computer calculations

$$d_0^{(p)} \equiv 1 = p^0$$

$$d_1^{(p)} = p = p^1$$

$$d_2^{(p)} = \frac{3}{2}p^2 + \frac{1}{2}p$$

$$d_3^{(p)} = \frac{5}{2}p^3 + 2p^2 + \frac{1}{2}p$$

$$d_4^{(p)} = \frac{35}{8}p^4 + \frac{71}{12}p^3 + \frac{25}{8}p^2 + \frac{7}{12}p$$

$$d_5^{(p)} = \frac{63}{8}p^5 + \frac{31}{2}p^4 + \frac{311}{24}p^3 + 5p^2 + \frac{2}{3}p$$

$$d_6^{(p)} = \frac{231}{16}p^6 + \frac{3043}{80}p^5 + \frac{2135}{48}p^4 + \frac{429}{16}p^3 + \frac{91}{12}p^2 + \frac{13}{20}p$$

Non-crossing partitions

Partial order on blocks

For $\pi \in \mathcal{NC}(n)$ with blocks $\pi = \{B_1, \ldots, B_k\}$ define *partial order*

$$B_i \preceq_\pi B_j \quad \text{if} \quad \min B_i \leq \min B_j \leq \max B_j \leq \max B_i$$

(block B_j is *inside* B_i).

Non-crossing partitions

Partial order on blocks

For $\pi \in \mathcal{NC}(n)$ with blocks $\pi = \{B_1, \ldots, B_k\}$ define *partial order*

$$B_i \preceq_\pi B_j \quad \text{if} \quad \min B_i \leq \min B_j \leq \max B_j \leq \max B_i$$

(block B_j is *inside* B_i). Since blocks are disjoint, we have either the strict order $B_i \prec_\pi B_j$ (i.e. $\min B_i < \min B_j$ and $\max B_j < \max B_i$) or $B_i = B_j$ i.e. $i = j$.

Non-crossing partitions

Partial order on blocks

For $\pi \in \mathcal{NC}(n)$ with blocks $\pi = \{B_1, \ldots, B_k\}$ define *partial order*

$$B_i \preceq_\pi B_j \quad \text{if} \quad \min B_i \leq \min B_j \leq \max B_j \leq \max B_i$$

(block B_j is *inside* B_i). Since blocks are disjoint, we have either the strict order $B_i \prec_\pi B_j$ (i.e. $\min B_i < \min B_j$ and $\max B_j < \max B_i$) or $B_i = B_j$ i.e. $i = j$.

Labels on blocks

For a partition $\pi = \{B_1, \ldots, B_k\} \in \mathcal{NC}(n)$ and the set $[p] := \{1, 2, \ldots, p\}$ consider the *label functions* $L : \pi \to [p]$ so that $L(B_j) \in [p]$ for each $1 \leq j \leq k$.

Weakly monotone ordered non-crossing pair partitions

Definition

We say that a non-crossing partition $\pi \in \mathcal{NC}(n)$ is *weakly monotone ordered by a label function* $L : \pi \to [p]$ if the function is *weakly monotone* with respect to the partial orders:

$$B_i \prec_\pi B_j \Longrightarrow L(B_i) \leq L(B_j)$$

Weakly monotone ordered non-crossing pair partitions

Definition

We say that a non-crossing partition $\pi \in \mathcal{NC}(n)$ is *weakly monotone ordered by a label function* $L : \pi \to [p]$ if the function is *weakly monotone* with respect to the partial orders:

$$B_i \prec_\pi B_j \Longrightarrow L(B_i) \leq L(B_j)$$

Notation: $\mathcal{NC}_2WMO([p], 2n)$

For positive integers $p, n \in \mathbb{N}$ we denote by $\mathcal{NC}_2WMO([p], 2n)$ the set of *all non-crossing pair partitions on* $[n]$ *weakly monotone ordered by label functions with values in*$[p]$ i.e. all pairs (π, L) where $\pi \in \mathcal{NC}_2(2n)$ and $L : \pi \to [p]$ is weakly monotone.

Combinatorial interpretation

Theorem: moments are the cardinalities

If $d_n^{(p)} := \varphi((G_1 + \cdots + G_p)^{2n})$ and $|\mathcal{N}C_2WMO([p], 2n)|$ is the cardinality, then

$$d_n^{(p)} = |\mathcal{N}C_2WMO([p], 2n)|.$$

Combinatorial interpretation

Theorem: moments are the cardinalities

If $d_n^{(p)} := \varphi((G_1 + \cdots + G_p)^{2n})$ and $|\mathcal{N}C_2WMO([p], 2n)|$ is the cardinality, then

$$d_n^{(p)} = |\mathcal{N}C_2WMO([p], 2n)|.$$

This gives the main tool to prove the recursion for the moments.