The analogue of the Hardy space H^1 for a complete Pick space

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H^2 and its relatives

The Hardy space is

$$H^2 = \left\{ f \in \mathcal{O}(\mathbb{D}) : \sup_{0<r<1} \int_0^{2\pi} |f(re^{it})|^2 \frac{dt}{2\pi} < \infty \right\}.$$

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$$H^\infty = \{ \varphi \in \mathcal{O}(\mathbb{D}) : \sup_{z \in \mathbb{D}} |\varphi(z)| < \infty \},$$

and

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The Smirnov class is

$$N^+ = \Big\{ \frac{\varphi}{\psi} : \varphi, \psi \in H^\infty, \overline{\psi H^2} = H^2 \Big\}.$$

Fact

$H^\infty \subset H^2 \subset H^1 \subset N^+$.

H^2 as an RKHS

H^2 is a reproducing kernel Hilbert space on $\mathbb{D}$, i.e. for $w \in \mathbb{D}$, there exists $k_w \in H^2$ with

$$\langle f, k_w \rangle = f(w) \quad (f \in H^2).$$

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Moreover,

$$H^\infty = \mathrm{Mult}(H^2) = \{\varphi : \mathbb{D} \to \mathbb{C} : \varphi \cdot f \in H^2 \text{ for all } f \in H^2\}$$

and

$$||\varphi||_\infty = ||\varphi||_{\mathrm{Mult}(H^2)}.$$

Goal

Find analogues of H^1 and N^+ for more general RKHS.

The Smirnov class

Let $\mathcal{H}$ be a RKHS of functions on a set X with kernel k.

Standing assumption

Assume k normalized at $x_0 \in X$, i.e. $k_{x_0}(x) = 1$ for all $x \in X$.

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$$N^+(\mathcal{H}) = \left\{ \frac{\varphi}{\psi} : \varphi, \psi \in \mathrm{Mult}(\mathcal{H}), \overline{\psi \mathcal{H}} = \mathcal{H} \right\}.$$

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Is $\mathcal{H} \subset N^+(\mathcal{H})$?

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Is $\mathcal{H} \subset N^+(\mathcal{H})$?

No, if $\mathcal{H} = L^2_a = \mathcal{O}(\mathbb{D}) \cap L^2(\mathbb{D})$.

Nevanlinna–Pick interpolation

Theorem (Pick 1916, Nevanlinna 1919)

Let $z_1, \dots, z_n \in \mathbb{D}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. There exists $f \in H^\infty$ with

$$f(z_i) = \lambda_i \text{ for } 1 \le i \le n \quad \text{and} \quad \|f\|_\infty \le 1$$

if and only if the matrix

$$\left[\frac{1 - \lambda_i \overline{\lambda_j}}{1 - z_i \overline{z_j}} \right]_{i,j=1}^n$$

is positive.

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if and only if the matrix

$$\Big[\frac{1 - \lambda_i \overline{\lambda_j}}{1 - z_i \overline{z_j}} \Big]_{i,j=1}^n = \Big[(1 - \lambda_i \overline{\lambda_j}) k_{z_j}(z_i) \Big]_{i,j=1}^n$$

is positive. Here $k_w(z) = (1 - z\overline{w})^{-1}$ is the reproducing kernel of H^2.

Complete Pick spaces

Let $\mathcal{H}$ be an RKHS on a set X with kernel k. Given $z_1, \ldots, z_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, does there exist $f \in \mathrm{Mult}(\mathcal{H})$ with

$$f(z_i) = \lambda_i \quad \text{for } 1 \le i \le n \quad \text{and} \quad \|f\|_{\mathrm{Mult}(\mathcal{H})} \le 1?$$

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$$f(z_i) = \lambda_i \quad \text{for } 1 \leq i \leq n \quad \text{and} \quad ||f||_{\mathrm{Mult}(\mathcal{H})} \leq 1?$$

A necessary condition is that the matrix

$$\left[k_{z_j}(z_i)(1 - \lambda_i \overline{\lambda_j}) \right]_{i,j=1}^n$$

is positive.

Definition

$\mathcal{H}$ is called a Pick space if this condition is sufficient. $\mathcal{H}$ is called a complete Pick space if the analogue of this condition for matrix valued functions is sufficient.

Examples

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  is a complete Pick space (Agler, 1988).
- The Drury-Arveson space H^2_d (a.k.a. symmetric Fock space) is the RKHS on $\mathbb{B}_d$, the open unit ball in $\mathbb{C}^d$, with kernel

$$k_w(z) = \frac{1}{1 - \langle z, w \rangle}.$$

  This is a complete Pick space.

The Smirnov class

Recall that

$$N^{+}(\mathcal{H}) = \left\{ \frac{\varphi}{\psi} : \varphi, \psi \in \mathrm{Mult}(\mathcal{H}), \overline{\psi \mathcal{H}} = \mathcal{H} \right\}.$$

Theorem (Aleman–H.–McCarthy–Richter)

Let $\mathcal{H}$ be a complete Pick space on X. Then $\mathcal{H} \subset N^{+}(\mathcal{H})$.

For the Drury-Arveson space H^2_d with $d < \infty$, this was shown by Alpay, Bolotnikov and Kaptanoğlu.

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Concrete formula

If $f \in \mathcal{H}$ with $||f|| = 1$, define $V_f(z) = 2\langle f, k_z f \rangle - 1$,

$$\psi = \frac{1}{V_f + 1}, \quad \varphi = \frac{f}{V_f + 1}.$$

Then $\varphi, \psi \in \mathrm{Mult}(\mathcal{H})$, $\overline{\psi \mathcal{H}} = \mathcal{H}$ and $f = \frac{\varphi}{\psi}$.

An application to zero sets

Let S be a set of functions on X. A subset $Z \subset X$ is called a zero set for S if there exists $f \in S$ with $Z = f^{-1}(0)$.

Corollary

Let $\mathcal{H}$ be a complete Pick space on X. Then the zero sets for $\mathcal{H}$ and for $\mathrm{Mult}(\mathcal{H})$ agree.

For the Dirichlet space, this is a theorem Marshall and Sundberg.

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Proof.

Since $\mathrm{Mult}(\mathcal{H}) \subset \mathcal{H}$, every zero set for $\mathrm{Mult}(\mathcal{H})$ is a zero set for $\mathcal{H}$. Conversely, let $f \in \mathcal{H}$. By the theorem, write $f = \frac{\varphi}{\psi}$ with $\varphi, \psi \in \mathrm{Mult}(\mathcal{H})$ and ψ non-vanishing. Then $f^{-1}(0) = \varphi^{-1}(0)$. □

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Definition

Let $\mathcal{H}$ be an RKHS. The weak product of $\mathcal{H}$ is

$$\mathcal{H} \odot \mathcal{H} = \Big\{ f = \sum_{n=0}^{\infty} g_n h_n : g_n, h_n \in \mathcal{H} \text{ and } \sum_{n=0}^{\infty} \|f_n\| \, \|g_n\| < \infty \Big\}.$$

Weak products and the Smirnov class

Recall that $H^1 \subset N^+$.

Theorem (Aleman–H.–M^{c}Carthy–Richter)

Let $\mathcal{H}$ be a complete Pick space that satisfies the column-row property. Then $\mathcal{H} \odot \mathcal{H} \subset N^+(\mathcal{H})$.

This applies to the Drury–Arveson space H^2_d for $d < \infty$ and the Dirichlet space.

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Theorem (Aleman–H.–M^c^Carthy–Richter)

Let $\mathcal{H}$ be a complete Pick space that satisfies the column-row property. Then multiplier invariant subspaces of $\mathcal{H}$ and of $\mathcal{H} \odot \mathcal{H}$ are in bijection via

$$\mathcal{M} \mapsto \overline{\mathcal{M}}^{\mathcal{H}\odot\mathcal{H}}$$
$$\mathcal{N} \cap \mathcal{H} \leftarrow \mathcal{N}.$$

Hankel forms on H^2

Definition

A Hankel form on H^2 with symbol $b \in H^2$ is a densely defined bilinear form

$$H_b : H^2 \times H^2 \to \mathbb{C}, \quad (\varphi, f) \mapsto \langle \varphi f, b \rangle \quad (\varphi \in H^\infty, f \in H^2).$$

Let $\mathcal{X} = \{ b \in H^2 : H_b \text{ is bounded} \}$.

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Theorem (Nehari, 1957)

$(H^1)^* = \mathcal{X}$.

Fefferman 1971: Function theoretic characterization of $\mathcal{X}$ as BMOA.

Hankel forms on complete Pick spaces

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Similar results for other classes of spaces:

- Coiffman–Rochberg-Weis (1976)
- Arcozzi–Rochberg-Sawyer–Wick (2010)
- Richter–Sundberg (2014).

Factorization in weak products

Theorem (Jury–Martin)

Let $\mathcal{H}$ be a complete Pick space that satisfies the column-row property. Then every $f \in \mathcal{H} \odot \mathcal{H}$ can be factored as $f = gh$ with $g, h \in \mathcal{H}$.

The column-row property

Definition

$\mathcal{H}$ has the column-row property if for all sequences (φ_n) in $\mathrm{Mult}(\mathcal{H})$,

$$\begin{bmatrix} M_{\varphi_1} \\ M_{\varphi_2} \\ \vdots \end{bmatrix} \text{ bounded} \quad \Rightarrow \quad \begin{bmatrix} M_{\varphi_1} & M_{\varphi_2} & \cdots \end{bmatrix} \text{ bounded.}$$

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Examples of spaces with the column-row property

- H^2 (easy)
- Dirichlet space (Trent)
- Drury–Arveson space H^2_d for $d < \infty$ (Aleman–H.–M^{c}Carthy–Richter).

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Question

Does every complete Pick space have the column-row property?

Thank you!