# $M$-ideals and split faces in matrix ordered spaces

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Joint work with
**Anil Kumar Karn**

13.12.2018

**This talk is based on the following papers:**

- **A. Ghatak**, A. Karn, *M-ideals and splits faces of the quasi state space of a non-unital ordered Banach space*, (To appear in **Positivity**.

- **A. Ghatak**, A. Karn, *CM-ideals and matricial split faces in ordered operator spaces*, (Under preparation).

# Outline

1. $M$-ideals in (non-unital) ordered Banach spaces
2. $CM$-ideals and split faces in ordered operator spaces

Let $V$ be a Banach space and let $W$ be a closed subspace of $V$.

**Annihilator** of $W$ : $\quad W^{\perp} = \{f \in V^* : f(x) = 0 \; \forall x \in W\}.$

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### Example

1. Let $K$ be a locally compact Hausdorff space. Then $W \subset C_0(K)$ is an $M$-ideal iff
$$W = \{f \in C_0(K) : f_{|_D} = 0\}$$
for some **closed set** $D \subset K$.

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   for some **closed set** $D \subset K$.
2. (Alfsen-Effros, 72) Let $\mathcal{A}$ be a $C^*$-algebra. Then $W$ is an $M$-ideal in $\mathcal{A}$ iff $W$ is closed **two sided ideal**.

# Split faces

- Let $F \subset K$ be two convex sets. Then $F$ is a *face* of $K$ if $u, v \in K$, and $\lambda \in (0,1)$ such that

$$\lambda u + (1-\lambda)v \in F \implies u, v \in F.$$

- Let $u \in K$. Then

$$\mathrm{face}_K(u) = \{v \in K : u = \lambda v + (1-\lambda)w, w \in K, \lambda \in (0,1)\}.$$

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- $F$ is a face , $F_K^C := \cup\{\mathrm{face}_K(v) : v \in K \text{ and } \mathrm{face}_K(v) \cap F = \emptyset\}.$

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- $F$ is a face , $F_K^C := \cup\{\text{face}_K(v) : v \in K \text{ and } \text{face}_K(v) \cap F = \emptyset\}.$

- A proper face $F \subset K$ is a **split face** of $K$ if $F_K^C$ is a proper face of $K$ such that $K = F \oplus_c F_K^C$. That is

$$v \in K \implies v = \lambda u + (1-\lambda)w$$

for **unique** $u \in F, w \in F_K^C$ and $\lambda \in [0,1]$.

# $M$-ideals in $A(K)$ spaces

Let $E$ be a locally convex space, and $K \subset E$ be a compact convex set. Let

$$A(K) = \{a : K \to \mathbb{R}, a \text{ is continuous and affine}\}.$$

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Let $W \subset A(K)$ be a closed subspace. Then $W$ is an $M$-**ideal** in $A(K)$ iff $W^{\perp} \cap K$ is a closed **split face** of $K$.

# Ordered Normed Spaces

Let $(V, V^+)$ be a real ordered vector space, $V^+$ is **proper** $(V^+ \cap -V^+ = 0)$, **generating** $\mathbf{(}V = V^+ - V^+\mathbf{)}$ and let $\|.\|$ be a norm on $V$ such that $V^+$ is closed.

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- $(O.\infty.1)$ If $u, v, w \in V$ with $u \leq v \leq w$, then

$$\|v\| \leq \max(\|u\|, \|w\|);$$

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$$\|v\| \leq \max(\|u\|, \|w\|);$$
- $(O.\infty.2)$ For $v \in V$ and $\epsilon > 0$, there exist $v_1, v_2 \in V^+$ such that
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$$v = v_1 - v_2 \text{ and } \max(\|v_1\|, \|v_2\|) \leq \|v\|.$$

## Examples of Order Smooth $\infty$-Normed Spaces

1. The self-adjoint part of $C^*$-algebra $\mathcal{A}_{sa}$;

2. An **order unit space** $(A, A^+, e)$

   $$[a \in A \implies -re \leq a \leq re, \quad \text{for some } r \geq 0];$$

3. An **approximate order unit space** $(A, A^+, \{e_\lambda\})$
   $[-re_\lambda \leq a \leq re_\lambda$ for some $e_\lambda, r \geq 0]$;

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   $[-re_\lambda \leq a \leq re_\lambda$ for some $e_\lambda, r \geq 0]$;
4. $A(K)$ with $A(K)^+ = \{a \in A(K) : a(k) \geq 0 \ \forall k \ \in K\}$.

## Order Smooth 1-Normed Space

Consider the following geometric conditions:

- $(O.1.1)$ If $u, v, w \in V$ with $u \leq v \leq w$, then
$$\|v\| \leq \|u\| + \|w\|;$$
- $(O.1.2)$ For $v \in V$ and $\epsilon > 0$, there are $v_1, v_2 \in V^+$ such that
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$(OS.1.2)$ For $v \in V$, there are $v_1, v_2 \in V^+$ such that

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### (Karn, 10)

Let $(V, V^+, \|.\|)$ be an ordered normed space, $V^+$ is proper, generating and normed closed.

Then $(O.\infty.1)$ on $V \iff (OS.1.2)$ on $V^*$.
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Hence $V$ is an order smooth $\infty$-normed space if and only if $V^*$ is an order smooth $1$-normed space satisfying the condition $(OS.1.2)$.

## Complementary set of a cone

Let $V$ be a normed linear space.

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- Let $V_1 = \{v \in V : \|v\| \leq 1\}$ and $v \in V$,

$$C(v) = \begin{cases} \text{cone}(\text{face}_{V_1}(\frac{v}{\|v\|})) & \text{if } v \neq \mathit{0} \\ 0 & \text{otherwise} \end{cases}$$

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- For a cone $C$ in $V$, we write

$$C' := \{v \in V : C \cap C(v) = \{0\}\}.$$

**(Alfsen-Effros, 72)**

Let $V$ be a Banach space, and let $W$ be a closed subspace of $V$. Then $W$ is an $M$-ideal in $V$ if and only if $W^{\perp'}$ is convex.

# Motivation

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**Question**

Let $(V, V^+, \|.\|)$ be an order smooth $\infty$-normed space, and let $W$ be closed subspace of $V$. Does $W^{\perp +'}$ is convex $\implies$ $W$ is an $M$-ideal in $V$.

# Technique

(G.-Karn, 18)

Let $(V, V^+, \|.\|)$ be a complete order smooth 1-normed space satisfying $(OS.1.2)$. Then

1. If $u \in V^+$, then
$$\mathrm{face}_{V_1}\left(\frac{u}{||u||}\right) = \mathrm{face}_{V_1^+}\left(\frac{u}{||u||}\right) \quad (V_1^+ := V_1 \cap V^+)$$
2. $(-V^+)' = V^+, (V^+)' = -V^+$.

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**Theorem (G.-Karn, 18)**

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$W$ satisfies the following conditions:

1. $W^{\perp'+}$ is convex.
2. $V^{*+} = W^{\perp+} \oplus_1 W^{\perp'+}$.

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Then $G$ is a **split face** of $Q(V)$ if

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Then $G$ is a **split face** of $Q(V)$ if $G'_{Q(V)}$ is also a face of $Q(V)$ and if every element in $Q(V)$ has a unique representation in $G \oplus_{c,1} G'_{Q(V)}$,

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## Theorem (G.-Karn, 18)

Let $V$ be a complete order smooth $\infty$-normed space and let $W$ be a closed subspace of $V$. Then $W$ is an $M$-**ideal** in $V$ iff $W^\perp \cap Q(V)$ is a **split face** of $Q(V)$.

# $M$-ideals and adjoining an order unit

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**Theorem (G.-Karn, 18)**

Let $V$ be a complete order smooth $\infty$-normed space. Then $V$ is an $M$-**ideal** in $\tilde{V}$ iff $V$ is an **approximate order unit space**.

$CM$-**ideals in matrix ordered spaces**

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(ii) $\|v \oplus w\|_{m+n} = \max\{\|v\|, \|w\|\}$ for all $v \in M_m(V), w \in M_n(V)$. ($L^\infty$**-condition**)

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Let $V$ be an operator space. Then its **matrix dual** $V^*$ is a **matrix normed space**, and thus $(M_n(V^*), \|.\|_n)$ is a Banach space for each $n$, where the action is given by

$$\langle [v_{i,j}], [f_{i,j}] \rangle = \sum_{i,j=1}^{n} f_{i,j}(v_{i,j})$$

for all $[v_{i,j}] \in M_n(V), [f_{i,j}] \in M_n(V^*)$.

Let $V$ be an operator space.

• Then its **matricial dual** $(V^*, \{\|.\|_n\})$ is an $L^1$-matrically normed space, i.e.

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(ii) $\|v \oplus w\|_{m+n} = \|v\|_m + \|w\|_n$ for all $v \in M_m(V), w \in M_n(V)$.
($L^1$-condition).

• Then its **matricial double dual** $V^{**}$ is an **operator space**.

A projection $P$ of an operator space $V$ is called a $CM$-**projection** if

$$\|v\|_n = \max\{\|P_n(v)\|_n, \|(I-P)_n(v)\|_n\} \ \ \forall v \in M_n(V).$$

Let $V$ be an operator space and $W$ be a closed subspace of $V$. Then $W$ is called a $CM$-**summand** if $W = P(V)$ for some $CM$-**projection** $P$ **of** $V$.

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**Definition (Effros-Ruan, 93)**

Let $V$ be an operator space and $W$ be a closed subspace of $V$. Then $W$ is called a $CM$-**ideal** in $V$ if $W^{\perp\perp}$ is a $CM$-summand in $V^{**}$.

**(Effros-Ruan, 93)**

Let $V$ be an operator space and let $W$ be a closed subspace of $V$. Then following are equivalent:

1. $W$ is a $CM$-ideal in $V$;
2. $M_n(W)$ is an $M$-ideal in $M_n(V)$ for each $n$;
3. $M_n(W^{\perp\perp})$ is an $M$-summand in $M_n(V^{**})$ for each $n$.

**Theorem (G.- Karn)**

Let $(V, \{\|\cdot\|_n\})$ be an operator space and let $W$ be a closed subspace of $V$. Then $W$ is a $CM$-**ideal** in $V$ if and only if there exist a $CL$-**projection** from $V^*$ onto $W^\perp$.

Let $V$ be a complex $*$-vector space. Then $V$ is called **matrix ordered space** if there is a cone $M_n(V)^+ \subset M_n(V)_{sa}$ for each $n$ such that $\gamma^* M_m(V)^+ \gamma \subset M_n(V)^+$ whenever $\gamma \in \mathbb{M}_{m,n}$.

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Let $(V, \{M_n(V)^+\}, \{\|.\|_n\})$ be a **matrix order operator space**. Then $V$ is called **matricially order smooth $\infty$-normed space** if $M_n(V)_{sa}$ is an order smooth $\infty$-normed space for each $n$.

Let $V$ be a complex $*$-vector space. Then $V$ is called **matrix ordered space** if there is a cone $M_n(V)^+ \subset M_n(V)_{sa}$ for each $n$ such that $\gamma^* M_m(V)^+ \gamma \subset M_n(V)^+$ whenever $\gamma \in \mathbb{M}_{m,n}$.

Let $(V, \{M_n(V)^+\}, \{\|.\|_n\})$ be a **matrix order operator space**. Then $V$ is called **matricially order smooth $\infty$-normed space** if $M_n(V)_{sa}$ is an order smooth $\infty$-normed space for each $n$.

**Theorem (G.- Karn)**

Let $(V, \{\| \cdot \|_n\}, \{M_n(V)^+\})$ be a matricially order smooth $\infty$-normed space and let $W$ be a closed **self-adjoint subspace** of $V$.
Then $W$ is a $CM$-**ideal** in $V$ if and only if $M_n(W)_{sa}$ is an $M$-ideal in $M_n(V)_{sa}$ for each $n \in \mathbb{N}$.

**Definition (G.-Karn)**

Let $V$ be a matricially order smooth $\infty$-normed space. Then an $L^1$-matricial convex set $\{D_n\}$ is called an **$L^1$-matricial split face** of $\{Q_n(V)\}$ if $D_n$ is a split face of $Q_n(V)$ for each $n$.

# $CM$-ideals and $L^1$-matricial split faces

**Definition (G.-Karn)**

Let $V$ be a matricially order smooth $\infty$-normed space. Then an $L^1$-matricial convex set $\{D_n\}$ is called an $L^1$-**matricial split face** of $\{Q_n(V)\}$ if $D_n$ is a split face of $Q_n(V)$ for each $n$.

**Theorem (G.-Karn)**

Let $V$ be a matricially order smooth $\infty$-normed space, and $W$ be a self-adjoint subspace of $V$. If $W$ is an $CM$-**ideal** in $V$ iff

1. $\{M_n(W^\perp) \cap Q_n(V)\}$ is an $L^1$-matrix convex set of $V^*$;
2. $M_n(W^\perp) \cap Q_n(V)$ is a split face of $Q_n(V)$ for each $n$.

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THANK YOU!