On analytic Chevalley-Shephard-Todd Theorem

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Motivation

- Let $\mathfrak{S}_n$ be the symmetric group on n symbols. Let us define the action of $\mathfrak{S}_n$ on $\mathbb{C}^n$ by $\boldsymbol{\sigma} \cdot \boldsymbol{z} = \left(z_{\boldsymbol{\sigma}^{-1}(1)}, \ldots, z_{\boldsymbol{\sigma}^{-1}(n)}\right)$ for $\boldsymbol{\sigma} \in \mathfrak{S}_n$ and $\boldsymbol{z} \in \mathbb{C}^n$.
- Consider a polynomial $f : \mathbb{C}^n \longrightarrow \mathbb{C}$. Let $\mathfrak{S}_n$ act on f by $\sigma(f)(z) = f(\sigma^{-1} \cdot z)$ for $\sigma \in \mathfrak{S}_n$ and $z \in \mathbb{C}^n$.
- $\mathbb{C}[z_1, \ldots, z_n]^{\mathfrak{S}_n} := \{f \in \mathbb{C}[z_1, \ldots, z_n] : \sigma(f) = f \text{ for all } \sigma \in \mathfrak{S}_n\}$.
- Note that

$$s_i(z) = \sum_{1 \leq k_1 < k_2 < \ldots < k_i \leq n} z_{k_1} \cdots z_{k_i},$$

is the elementary symmetric polynomial of degree i in n variables.

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Proposition

$\mathbb{C}[z_1,\ldots,z_n]^{\mathfrak{S}_n} = \mathbb{C}[s_1,\ldots,s_n]$.

Proposition

$\mathbb{C}[z_1,\ldots,z_n]$ is a free module over the ring of symmetric polynomials, $\mathbb{C}[s_1,\ldots,s_n]$ of rank $n!$.

For n=2

Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a polynomial map. Note that polynomial f can be decomposed as follows:

$$f(z_1, z_2) = \frac{f(z_1, z_2) + f(z_2, z_1)}{2} + \frac{f(z_1, z_2) - f(z_2, z_1)}{2}. \tag{0.1}$$

Let $g(z_1, z_2) = \frac{f(z_1,z_2) - f(z_2,z_1)}{2}$. Clearly, $g(z_1, z_2) = -g(z_2, z_1)$, that is, g is an anti-symmetric function.

Thus there exists a symmetric polynomial h such that $g(z_1, z_2) = (z_1 - z_2)h(z_1, z_2)$.

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Equality (0.1) can be re-written as

$$f(z_1,z_2) = \frac{f(z_1,z_2)+f(z_2,z_1)}{2} + (z_1 - z_2)h(z_1,z_2).$$

Thus any element of $\mathbb{C}[z_1,z_2]$ can be expressed as a linear sum of 1 and $z_1 - z_2$ over the ring of symmetric polynomials $\mathbb{C}[z_1,z_2]^{\mathfrak{S}_2}$.

In other words, $\mathbb{C}[z_1,z_2]$ is a free module of rank 2 over $\mathbb{C}[z_1,z_2]^{\mathfrak{S}_2}$ with $\{1, z_1 - z_2\}$ as one choice of basis.

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Questions:

- For which kind of groups, we can expect a similar result?
- How the rank of the module is related to the order of the group?
- What happens if we replace polynomials by holomorphic functions?

Pseudo-reflection Group

Definition

A pseudo-reflection on $\mathbb{C}^n$ is a linear endomorphism $\rho : \mathbb{C}^n \to \mathbb{C}^n$ such that the rank of $1-\rho$ is 1. Equivalently, ρ is not the identity map and it fixes a hyperplane pointwise.

Let G be a finite group generated by pseudo-reflections. Then G also acts on the set of functions on $\mathbb{C}^n$ by $\rho(f)(z) = f(\rho^{-1} \cdot z)$.

Let $A = \mathbb{C}[z_1, \ldots, z_n]$ be the ring of polynomial functions on $\mathbb{C}^n$ and let $B = A^G$ be the ring of G-invariant elements of A.

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Chevalley-Shephard-Todd Theorem

Theorem

$B = \mathbb{C}[\theta_1, \ldots, \theta_n]$, where θ_i are algebraically independent homogeneous polynomials, that is, B is itself a polynomial algebra in n variables.

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A is a free B module of rank d, where d is the order of G. Further, one can choose a basis of A consisting of homogeneous polynomials.

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Next we want to generalize Chevalley-Shephard-Todd theorem to the setting of holomorphic functions on $\mathbb{C}^n$.

Let $\theta : \mathbb{C}^n \to \mathbb{C}^n$ be the function defined by

$$\theta(z) = \big(\theta_1(z), \ldots, \theta_n(z)\big),\ z \in \mathbb{C}^n.$$

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Analytic CST Theorem

Theorem

For a G-invariant holomorphic function f on $\mathbb{C}^n$, there exists a unique holomorphic function g on $\mathbb{C}^n$ such that $f = g \circ \theta$.

Let $q_1, \ldots, q_d$ (where $d = |G|$) be a basis of A as a B module.

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Some discussion

- Let the polynomials $q_1, q_2, \cdots, q_d$ form a basis of A as a B module.
- Let us take a holomorphic function $f : \mathbb{C}^n \longrightarrow \mathbb{C}$. Then the previous theorem ensures that there exist G-invariant holomorphic functions $f_1, f_2, \cdots, f_d$ such that

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- Let $G = \{\rho_1, \ldots, \rho_d\}$.
  Applying ρ_i to the above equation gives

$$\rho_i(f) = f_1 \rho_i(q_1) + \cdots + f_d \rho_i(q_d), 1 \leq i \leq d.$$

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- Let $\boldsymbol{y}$ be the column vector $(f_1,\ldots,f_d)^t$ and let $\boldsymbol{x} = \big(\rho_1(f),\ldots,\rho_d(f)\big)^t$.
- Note that $M\boldsymbol{y} = \boldsymbol{x}$ and hence

$$(\det M)\boldsymbol{y} = (\text{adj M})\boldsymbol{x},$$

where

$$M = \Big(\!\Big(\rho_i(q_j)\Big)\!\Big)_{i,j=1}^{d}.$$

- Next lemma ensures unique solution of the above system of equations.

Lemma

$\det M$ *is not identically zero.*

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Determinantal formula for M

- We call a hyperplane H of $\mathbb{C}^n$ *reflecting* if there exists a pseudo-reflection in G fixes H pointwise.
- Let $H_1, \ldots, H_t$ be the distinct reflecting hyperplanes associated to G.
- Let K_i be the subgroup generated by the pseudo-reflections of G those fix H_i pointwise.
- Let $m_1, \ldots, m_t$ be the orders of the corresponding subgroups $K_1, \ldots, K_t$.
- Let L_i be the nonzero linear function on $\mathbb{C}^n$ defining H_i, that is, $H_i = \{z \in \mathbb{C}^n : L_i(z) = 0\}$.

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Determinantal formula for M

Proposition

$$\det(M) = c\prod_{i=1}^{t} L_i^{d(m_i-1)/2},$$

where c is a nonzero constant.

On G-invariant domain

Theorem

Let G be a finite pseudo-reflection group. For a G-invariant holomorphic function f on a G-invariant domain Ω, there exists a unique holomorphic function g on $\theta(\Omega)$ such that $f = g \circ \theta$, where $\theta = (\theta_1, \ldots, \theta_n)$.

Let $q_1, \ldots, q_d$ (recall that $d = |G|$) be a basis of A as a B module.

Theorem

Let f be a holomorphic function on a G-invariant domain Ω. Then there exist unique G-invariant holomorphic functions $f_1, \ldots, f_d$ such that $f = f_1 q_1 + \cdots + f_d q_d$.

Thank You!