Spectrum of random Schrödinger operators with decaying randomness

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The Model

- $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$, d dimensional Laplacian,
- V^ω is the multiplication operator on $L^2(\mathbb{R}^d)$,

$$(V^\omega f)(x) = V^\omega(x) f(x), \;\; V^\omega(x) = Q(x) \sum_{n \in \mathbb{Z}^d} \omega_n \chi_{n+(0,1]^d}(x).$$

  $Q(x) = O(|x|^{-\alpha}), \;\; \alpha > 0$ for large x and $\{\omega_n\}_n$ are iid random variables with common distribution by μ, $\frac{d\mu}{dx}(x) = O(|x|^{-(1+\delta)}), \;\; \delta > 0, \;\; |x| \to \infty$.
- Consider the probability space $\left(\Omega = \mathbb{R}^{\mathbb{Z}^d}, \; \mathcal{B}_\Omega, \; \mathbb{P} = \otimes \mu\right)$. Define the random operator H^ω as

$$H^\omega = -\Delta + V^\omega, \; \omega = (\omega_n)_{n \in \mathbb{Z}^d} \in \Omega.$$

The spectrum of $-\Delta$

- It is well known that $-\Delta$ is essential self-adjoint and

$$\mathcal{F}(-\Delta)\mathcal{F}^{-1} = M_{\varphi(x)},\ \varphi(x) = \sum_{i=1}^{d} |x_i|^2,\ x \in \mathbb{R}^d.$$

  $\mathcal{F}$ is the Fourier transform on $L^2(\mathbb{R}^d)$.
- Now we have $\sigma(-\Delta) = \sigma_{ac}(-\Delta) = [0, \infty)$.
- Let $-\Delta_L$ is the restriction of $-\Delta$ to the domain $(-L, L)^d$ with Neumann boundary condition.

$$\sigma(-\Delta_L) = \sigma_{dis}(-\Delta_L) = \left\{ \left(\frac{\pi}{2L}\right)^2 \sum_{i=1}^{d} n_i^2 :\ n_i \in \mathbb{N} \cup \{0\} \right\}.$$

Results (Spectrum of H^ω)

- For $\alpha\delta \leq d$ we have $\sigma(H^\omega) = \sigma_{ess}(H^\omega) = \mathbb{R}$ a.e ω.
- For $\alpha\delta > d$ we have $\sigma_{ess}(H^\omega) = [0, \infty)$ and $\sigma(H^\omega) \cap (-\infty, 0)$ is discrete a.e ω.
  In above case 0 may be the limit point for negative eigenvalues. But for $(\alpha - 2)\delta > d$ we have $\#\{\sigma(H^\omega) \cap (-\infty, 0)\} < \infty$.
- For $\delta > 2$ and $\alpha > 1$ we have $[0, \infty) \subset \sigma_{ac}(H^\omega)$ a.e ω.

The negative spectrum (Anderson localization)

- The negative spectrum of H^ω always exhibits exponential localization (Anderson Localization), independent of the choice of α and δ.
- The negative part of the spectrum always pure point i.e $(-\infty, 0) \cap \sigma(H^\omega) \subset \sigma_{pp}(H^\omega),\ a.e\ \omega.$

$$H^\omega \psi_\omega = E\psi_\omega,\ \psi_\omega(x) \leq c_\omega e^{-d_\omega |x-\eta_\omega|}, E < 0,\ a.e\ \omega.$$

η_ω is the localization center, ψ_ω attain its maximum at η_ω.

Out line of the proof

- Using Weyl's criterion together with Borel-Cantelli lemma we get (for any choice of α and δ)

$$[0, \infty) \subset \sigma_{ess}(H^{\omega}), \ \ a.e\, \omega.$$

- For $\alpha\delta > d$ the Dirichlet Neumann bracketing $\left(\oplus_{n\in\mathbb{Z}^d} H^{\omega}_{n,N} \leq H^{\omega} \leq \oplus_{n\in\mathbb{Z}^d} H^{\omega}_{n,D}\right)$ will give

$$\#\{(-\infty, -\epsilon) \cap \sigma(H^{\omega})\} < \infty, \ \ a.e\, \omega, \ \forall\, \epsilon > 0.$$

- For $\alpha\delta > d$ we have $[0, \infty)$ is the essential spectrum and below zero there is the discrete spectrum a.e ω.

- For $\alpha\delta > d$ still 0 may be the limit point for the negatives eigenvalues.
- But for $(\alpha - 2)\delta > d$ we can show

$$H^{\omega} \geq -\Delta - \frac{M^{\omega}}{1 + |x|^{\epsilon}}, \ \epsilon > 2, \ a.e\, \omega.$$

- Let $H = -\Delta - V$ with $V(x) = O(|x|^{\epsilon}), \ \epsilon > 2$. The number of negative eigenvalues of H is finite.
- Now we get

$$\#\{(-\infty, 0) \cap \sigma(H^{\omega})\} < \infty, \ a.e\, \omega.$$

- Using min-max principle we can show that

$$\bigcup_{\lambda\in\mathbb{R}} \sigma\big(-\Delta + \lambda\chi_{(0,1]^d}\big) = \mathbb{R}.$$

- For $\alpha\delta \leq d$

$$\bigcup_{\lambda\in\mathbb{R}} \sigma\big(-\Delta + \lambda\chi_{(0,1]^d}\big) \subseteq \sigma_{ess}(H^\omega),\ a.e\ \omega.$$

- The above two will imply

$$\sigma(H^\omega) = \sigma_{ess}(H^\omega) = \mathbb{R},\ a.e\ \omega,\ for\ \alpha\delta \leq d.$$

Absolutely continuous spectrum

- If the potential decay fast enough, $\delta > 2$ and $\alpha > 1$ then we verified the following:

$$\int_{\mathbb{R}^d} (1+|x|)^{-2m} \big(V^{\omega}(x)\big)^2 dx < \infty, \;\; a.e\, \omega, \;\; for\ some\ m > 0,$$

$$\int_1^{\infty} \left(\int_{a<|x|<v} \big(V^{\omega}(xt)\big)^2 dx \right) dt < \infty, \; a.e\, \omega, \; 0 < a < b < \infty.$$

- With above two estimation (Cook's Method, scattering theory, existence of wave operators) will ensure that $[0,\infty) \subset \sigma_{ac}(H^{\omega}), \;\; a.e\, \omega.$

Negative part of the spectrum (Wegner estimate)

- Let $H^{\omega}_{\Lambda_L(x)}$ be the restriction of H^{ω} to the cube $\Lambda_L(x)$ with center at x and side length L. Set

$$\Omega_L = \{\omega : |V^{\omega}(n)| < L^a,\ n \in \Lambda_L(0)\}, a > 0,\ |V^{\omega}(n)| \simeq \frac{\omega_n}{|n|^{\alpha}}.$$

- Wegner estimate for $E < 0$

$$\sup_{n \in \mathbb{Z}^d} \mathbb{P}\left(dist\left(\sigma\left(H^{\omega}_{\Lambda_L(n)}\right), E\right) < \eta \,\middle|\, \Omega_L \right) \leq C\, \eta^s L^{d+\gamma a}.$$

- The above estimate follows from

$$\mathbb{E}\left(Tr\left(E_{H^{\omega}_{\Lambda_L(n)}}(I)\right)\right) \leq C\, |I|^s L^{d+\gamma a},\ C, \gamma, a > 0,\ s \in (0, 1].$$

Initial Scale estimate

- The Initial scale estimate for $E < 0$ is given by $(c, m, b > 0)$

$$\mathbb{P}\left(\left\|\chi_{\partial\Lambda_L}\left(H^{\omega}_{\Lambda_L(n)} - E\right)^{-1}\chi_{\Lambda_{\frac{L}{3}}(0)}\right\| \leq ce^{-mL}\right) \geq 1 - \frac{1}{L^b}.$$

- Once we have Wegner estimate and Initial scale estimate we can use Bootstrap Multiscale analysis (Germinet-Klein) and show that $(-\infty, 0)$ exhibits exponential localization.
- This Bootstrap Multiscale analysis is an induction method and to start the induction all we need the Wegner estimate and the Initial scale estimate.

- We can see that H^ω is densely define with domain $C_c^\infty(\mathbb{R}^d)$ a.e ω.
- For $(2+\alpha)\delta > d$ we have essential self-adjointness of H^ω.
- The above choice of α and δ we can show that

$$V_-^\omega(x) \leq M^\omega(1+|x|)^{2-\epsilon}, \ \epsilon > 0, \ a.e\, \omega.$$

- It is known that if $V_-(x) = o(|x|^{2-\epsilon})$ then $-\Delta + V$ is essential self-adjoint on $L^2(\mathbb{R}^d)$. Here $V_-(x) = min\{0, V(x)\}$.

The reason to study the spectrum of H^ω

- In Mathematical Physics there is a phenomenal called existence of extended states in low disorder.
- Define H^ω_λ on $L^2(\mathbb{R}^d)$ by

$$H^\omega_\lambda = -\Delta + \lambda \sum_{n\in\mathbb{Z}^d} \omega_n\, u(x-n),$$

  u is compactly supported and $u \in L^\infty(\mathbb{R}^d)$, $\{\omega_n\}$ are iid random variables and $\lambda > 0$.
- It is expected that for small enough λ

$$\emptyset \neq \sigma_{ac}(H^\omega_\lambda) \subset [0,\infty),\ \ a.e\ \omega.$$

Reference

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Thank You