E_0-semigroups arising from boundary weight maps

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CP-semigroups and E_0-semigroups

Definition

A CP-semigroup is a family $\{\alpha_t : t \geq 0\}$ of normal contractive completely positive maps of $B(H)$ such that:

- $\alpha_t \circ \alpha_s = \alpha_{t+s}$, for all $t, s \geq 0$;
- $\alpha_0(X) = X$, for all $X \in B(H)$;
- the map $t \mapsto \langle \alpha_t(X)\xi, \eta \rangle$ is continuous, for all $\xi, \eta \in H$, $X \in B(H)$

It is an E_0-semigroup if α_t is a unital *-endomorphism for all t.

Example (CP semigroup on $M_2(\mathbb{C})$)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & e^{-t}b \\ e^{-t}c & d \end{bmatrix}$$

Definition

Two E_0-semigroups α, β on $B(H)$ are cocycle equivalent if there exists $(U_t)_{t\geq 0}$ strongly continuous family of unitary operators,

$$U_{t+s} = U_t \alpha_t(U_s), \qquad t, s \geq 0$$
$$\beta_t(X) = U_t \alpha_t(X) U_t^*.$$

They are cocycle conjugate ($\alpha \sim \beta$) if there exists a conjugacy θ such that α and $\theta^{-1} \circ \beta \circ \theta$ are cocycle equivalent.

B.V. Rajarama Bhat's Dilation Theorem

Let α be a unital CP-semigroup on $B(K)$. Then there exists an E_0-semigroup α^d on $B(H)$ and an isometry $W : K \to H$ such that $\alpha_t^d(WW^*) \geq WW^*$ and $\alpha_t(A) = W^* \alpha_t^d(WAW^*)W$ for all $A \in B(K)$ and $t \geq 0$. The dilation α^d can be taken to be *minimal* in the appropriate sense. In this case α^d is unique up to cocycle conjugacy.

Definition

A unit for an E_0-semigroup α on $B(H)$ is a strongly continuous one-parameter semigroup of operators T_t such that $T_0 = I$ and

$$\alpha_t(X)T_t = T_tX, \qquad \forall t \geq 0, \forall X \in B(H).$$

An E_0-semigroup is called:

- spatial if it has a unit
- type I if it is generated by its units
- type II for all other spatial semigroups
- non-spatial or type III if it has no units.

Definition (Powers '88 + Arveson'89)

The index invariant counts the relative abundance of units of spatial E_0-semigroups. It has values in $\{0, 1, 2, \ldots, \infty = \aleph_0\}$.

Today's question

Arveson completely classified type I by the index. They are cocycle conjugate to CAR/CCR flows.

General Problem

Find a rich and interesting class of E_0-semigroups (beyond type I) which can be classified effectively up to cocycle conjugacy.

Today's goal

We will describe the classification of the class of E_0-semigroups of type II and index zero (denoted II_0) which arise from q-pure q-weight maps over finite dimensional spaces.

JMP'18 C. Jankowski, D. Markiewicz and R.T. Powers, *"Classification of q-pure q-weight maps over finite dimensional Hilbert spaces"*, arXiv:1807.09824 [math.OA].

CP-flows Approach

Definition (Powers '03)

Let K be a separable Hilbert space, and let $H = L^2(0, \infty; K)$. We denote by S_t the right shift of H by $t \geq 0$.

A CP-flow over K is a CP semigroup α of $B(H)$ such that for all $t \geq 0$ and $A \in B(H)$ we have

$$\alpha_t(A)S_t = S_t A.$$

Theorem (Powers '03)

Every unital CP-flow α has a minimal dilation E_0-semigroup α^d which is also a CP-flow. We call α^d the minimal flow dilation of α.

Furthermore, every spatial E_0-semigroup is cocycle conjugate to an E_0-semigroup which is a CP-flow.

Given two CP-semigroups α, β maps on $B(H)$, we will denote

$$\alpha \geq \beta \quad \iff \quad \alpha_t - \beta_t \text{ is CP for all } t \geq 0$$

Definition (JMP'12)

A CP-flow α is called q-pure if its set of CP-flow subordinates is totally ordered. In other words, if β and γ are CP-flows such that $\alpha \geq \beta$ and $\alpha \geq \gamma$, then either $\beta \geq \gamma$ or $\gamma \geq \beta$.

It is not obvious that q-purity is a cocycle conjugacy invariant property. However, a unital CP-flow which has type II$_0$ is q-pure if and only if it is aligned, and the latter is a cocycle conjugacy invariant (see JMP'15).

Constructing CP-flows - basic definitions

Let K be a separable Hilbert space (possibly $\dim K < \infty$), let $H = K \otimes L^2(0, \infty)$.

- $\Lambda : B(K) \to B(H)$ is given by $(\Lambda(T)f)(x) = e^{-x}Tf(x)$.
- The null boundary algebra is

$$\mathfrak{A}(H) = (I - \Lambda(I_K))^{1/2} B(H) (I - \Lambda(I_K))^{1/2}$$

- Set $\mathfrak{A}(H)_*$ of boundary weights or b-weights
  $=$ set of linear functionals $\mu : \mathfrak{A}(H) \to \mathbb{C}$ such that

$$A \mapsto \mu\Big([I - \Lambda(I_K)]^{\frac{1}{2}} A [I - \Lambda(I_K)]^{\frac{1}{2}} \Big)$$

  is a normal bounded linear functional on $B(H)$.

Example (b-weight for $K = \mathbb{C}$, $H = L^2(0, \infty)$)

$\mu(A) = \langle f, Af \rangle$ where $f(x) = e^{-x}/\sqrt{2x}$.

Definition (Powers '03, but dualized as in JMP'18)

We say that $\omega : \mathfrak{A}(H) \to B(K)$ is a q-weight map over K if

- ω is completely positive
- $\omega(I - \Lambda(I_K)) \leq I_K$
- $\pi_t = (I + \omega_t \Lambda)^{-1} \omega_t$ exists and it is c.c.p. for all $t > 0$, where $\omega_t(A) = \omega(S_t S_t^* A S_t S_t^*)$.

Range rank of ω $= \dim \mathrm{Ran}(\omega)$.

Example ($K = \mathbb{C}$ and so range rank one)

$\omega(A) = \langle f, Af \rangle\, I$ where $f(x) = e^{-x}/\sqrt{2x}$.

Theorem (Powers '03)

There exists a 1-1 correspondence between CP-flows over K and q-weight maps over K. A CP-flow is unital if and only if its associated q-weight map satisfies $\omega(I - \Lambda(I_K)) = I_K$.

Survey of results about CP-flows over K

- $\dim K = 1$ and type II_0 (automatic range rank one):
  - Powers '03: classification up to cocycle conjugacy.

- $1 < \dim K < \infty$ and type II_0
  - Jankowski '10: Study of class of *boundary weight doubles*. Classification in some special cases which are q-pure and range rank one or max range rank.
  - JMP'12: Suppose that ω_1 and ω_2 are q-pure q-weight maps of type II_0 over finite dimensional K_1 and K_2. If $\dim K_1 \neq \dim K_2$, then the E_0 semigroups are not cocycle conjugate. And in fact for each value of $\dim K < \infty$ there are uncountably many.

- $1 < \dim K$ and type II_0
  - JMP'12: Suppose that ω is a q-weight map over K of type II_0 that is q-pure. Then ω cannot have range rank 2. If $\dim K = 2$, the range rank of ω must be 1 or 4.

q-pure q-weight maps of range rank 1

In fact, in JMP'12 we obtained a classification of q-pure q-weight maps of range rank one.

Theorem (JMP'12– Range rank one characterization)

Suppose ω is a range rank 1 q-weight map over K separable, i.e. $\omega(A) = \mu(A)T$ for $A \in \mathfrak{A}(H)$ where $0 \leq T$ has norm one and $\mu \in \mathfrak{A}(H)_*$ such that $\mu(I - \Lambda(T)) \leq 1$ and $\mu(I) = \infty$. Then ω is q-pure if and only if the following three conditions are met.

(i) *T is a projection.*

(ii) *μ is strictly infinite, i.e. it has no bounded subordinates*

(iii) *If $e \in B(K)$ is a rank one projection with $e \leq T$ then $\mu(\Lambda(e)) = \infty$.*

Cocycle conjugacy and q-purity: range rank one

Theorem (JMP'12– proof in finite dimensions, true in general)

For each $i = 1, 2$, suppose that ω_i is a q-pure range rank one q-weight over separable Hilbert spaces K_i, so that there exists a projection T_i and a q-pure $\mu_i \in \mathfrak{A}(K_i)_*$ such that $\omega_i(A) = \mu_i(A)T_i$ for all $A \in \mathfrak{A}(K_i)$.
Then the E_0-semigroups induced by ω_1 and ω_2 are cocycle conjugate if and only if there exists a partial isometry $U : K_1 \to K_2$ and $\lambda > 0$ so that $U^*U = T_1$, $UU^* = T_2$ and μ_1 and μ_2 can be expressed in the form

$$\mu_1(A) = \sum_{k \in J} (f_k, Af_k), \qquad \mu_2(B) = \sum_{k \in J} (g_k, Bg_k)$$

for $A \in \mathfrak{A}(K_1)$, $B \in \mathfrak{A}(K_2)$ with $g_k = \lambda(U \otimes I)f_k + h_k$ where $h_k \in K_1 \otimes L^2(0, \infty)$ for $k \in J$ and $\sum_{k \in J} \|h_k\|^2 < \infty$.

What about q-purity of range rank > 1?

Theorem (JMP'12– Boundary expectations for $\dim L < \infty$)

Suppose that ω is a q-weight map over K finite dimensional of type II$_0$. Then there exists a boundary expectation, i.e. a completely contractive linear idempotent $L : B(K) \to \mathrm{Ran}(\omega)$.

It follows that $\mathrm{Ran}(\omega)$ becomes a von Neumann algebra $\mathcal{L}$ with the Choi-Effros product $A \star B = L(AB)$ and the same linear and adjoint operations.

Let π_t be the generalized boundary representation. Any limit point L of $\pi_t\Lambda$ is a boundary expectation. Use compactness to obtain one.

It is unclear whether the $\mathrm{Ran}(\omega)$ has to be weak*-closed when $\dim K = \infty$. That occurs trivially when $\dim K < \infty$.

Choi-Effros Factors

Theorem (JMP'18)

Suppose that ω is a q-weight map over K finite dimensional of type II$_0$ with boundary expectation L and associated Choi-Effros algebra $\mathcal{L}$. If Q is a non-zero central projection in $\mathcal{L}$, then there is a q-subordinate η of ω such that $\mathrm{Ran}(\eta) = Q \star \mathcal{L}$. In particular, if ω is q-pure, then $\mathcal{L}$ is a Choi-Effros factor of type I$_q$ for $q \leq \dim K$.

Corollary

A q-pure q-weight map over K finite dim. cannot have range rank 2.

Obviously this is a much more conceptual reason for the JMP'12 result, albeit only for $\dim K < \infty$. It is tantalizing for $\dim K = \infty$. Note it also explains the JMP'12 result that for $\dim K = 2$, q-purity implies range rank one or four.

Characterization of q-purity

Under the condition that ω is q-weight over $\mathbb{C}^p$ of type II_0, with $\mathcal{L} = \mathrm{Ran}(\omega)$ is type I_q

- For a proper normalization, the map $(I + \omega_t \Lambda)|_{\mathcal{L}}$ converges to a CP limit with inverse ψ conditionally negative.
- Define $\vartheta = \psi\omega$, (which is an $\mathcal{L}$-valued boundary weight map) and get a decomposition $\omega = \psi^{-1}\vartheta$
- Obtain conditions so that a pair (ψ, ϑ) will be called $\mathcal{L}$-serendipitous iff it gives rise to a q-weight map as above with range $\mathcal{L}$.
- If (ψ', ϑ) is serendipitous, then $\omega' = \psi'^{-1}\vartheta$ is a q-subordinate of ω (i.e. $\omega' \leq_q \omega$) if and only if $\psi \leq \psi'$. This is inspired by Chris Jankowski's thesis.

Definition

A map $\phi : B(H) \to B(H)$ is conditionally positive if it is hermitian $(\phi(A^*) = \phi(A)^*$ for all A) and if $A_i \in B(H)$, $f_i \in H$ for $i = 1, 2, \ldots n$ satisfy

$$\sum_{i=1}^{N} A_i f_i = 0,$$

then

$$\sum_{i,j=1}^{n} (f_i \phi(A_i^* A_j) f_j) \geq 0$$

And ϕ is conditionally negative if $-\phi$ is conditionally positive.

Theorem (JMP'18)

Suppose that ω is a q-weight over K finite dim. of type II$_0$ and $\mathcal{L} = \mathrm{Ran}(\omega)$. Then ω is q-pure iff the following are satisfied:

- *$\mathcal{L}$ is a factor of type I$_q$ for $q \leq \dim K$, and in this case let $\omega = \psi^{-1}\vartheta$ be its serendipitous decomposition.*
- *The $\mathcal{L}$-valued boundary weight map ϑ is strictly infinite*
- *if $0 \neq e \in \mathcal{L}'$ is a projection and $e \leq I_{\mathcal{L}}$ the unit of $\mathcal{L}$, then $\|\omega(\Lambda(e))\| = \infty$.*

Now ϑ can be written in terms of matrix units coordinates in $\mathcal{L} \simeq M_q(\mathbb{C})$. Then we can write $\vartheta_{ij}(A) = \sum_{k \in J}(v_{ik}, Av_{jk})$. Part of the serendipity condition is that the v_{ij} are related and determined by v_{11}, but the serendipity condition is not easily stated in terms of v_{11} alone.

Theorem (JMP'12– Range rank one characterization)

Suppose ω is a q-weight map over K of the form $\omega(A) = \mu(A)T$ for $A \in \mathfrak{A}(H)$ where $0 \leq T$ has norm one and $\mu \in \mathfrak{A}(H)_*$ such that $\mu(I - \Lambda(T)) \leq 1$ and $\mu(I) = \infty$. Then ω is q-pure iff the following hold.

- *T is a projection.*
- *μ is strictly infinite, i.e. it has no bounded subordinates*
- *If $e \in B(K)$ is a rank one projection with $e \leq T$ then $\mu(\Lambda(e)) = \infty$.*

Theorem (JMP'18)

Suppose that ω is a q-weight over K finite dim. of type II$_0$, $\mathcal{L} = \mathrm{Ran}(\omega)$. Then ω is q-pure iff the following are satisfied:

- *$\mathcal{L}$ is a factor of type I$_q$ for $q \leq \dim K$, and in this case let $\omega = \psi^{-1}\vartheta$ be its serendipitous decomposition.*
- *The $\mathcal{L}$-valued boundary weight map ϑ is strictly infinite*
- *if $0 \neq e \in \mathcal{L}'$ is a projection and $e \leq I_{\mathcal{L}}$, then $\|\omega(\Lambda(e))\| = \infty$.*

Classification

Classification Theorem (JMP'18)

Suppose that ω is a q-weight map over $\mathbb{C}^{qm}$ of type II_0 with boundary expectation L and associated Choi-Effros algebra $\mathcal{L}$, which is a factor of type I_q. Then ω is cocycle conjugate to a unital q-pure q-weight map η over $\mathbb{C}^m$ of range rank one.

And we figured out how to describe and compare all range rank one, so in principle this tells you that for classification we know the situation when $\dim K < \infty$.

There is lot hiding in this result.

- The size of $\mathcal{L}$ divides the dimension of K.
- The construction of the rank one q-weight
- Freedom in this construction

Write $\omega = \psi^{-1}\vartheta$, $\vartheta_{ij}(A) = \sum_{k\in J}(v_{ik}, Av_{jk})$

$$\vartheta_{ij}(A) = \sum_{k\in J}((g_{ik} + h_{ik}), A(g_{jk} + h_{jk}))$$

for $A \in \mathfrak{A}(\mathbb{C}^p)$ where the $g_{ik},\ h_{ik} \in \mathbb{C}^p \otimes L^2_+(0,\infty)$ and

$$g_{ik}(x) = E_{i1}g_k(x), \qquad E_{11}g_k(x) = g_k(x), \qquad \sum_{i=1}^{q} E_{1i}h_{ik}(x) = 0$$

for $A \in \mathfrak{A}(\mathbb{C}^p)$, $x \geq 0$, $i, j \in \{1, \cdots, q\}$ and $k \in J$ a countable index set and the $h_{ik} \in \mathbb{C}^p \otimes L^2(0,\infty)$ and if

$$w_t = \sum_{k\in J}(g_k, \Lambda|_t g_k) \qquad \rho_{ij}(A) = \sum_{k\in J}(h_{ik}, Ah_{jk})$$

then ρ is bounded so

$$\sum_{k\in J}\|h_{ik}\|^2 < \infty \qquad \text{and} \qquad \sum_{k\in J}(g_k, (I - \Lambda)g_k) < \infty$$

and $1/w_t \to 0$ as $t \to 0+$ and ψ satisfies the conditions

$$\psi(I_o) \geq \vartheta(I - \Lambda(I_o)) \qquad \text{and} \qquad \psi + \rho\tilde{\Lambda} \text{ is cond. negative}$$

We show that ω is cocycle conjugate to $\eta = s_0 I_m \mu$ where

$$\mu(A) = \sum_{k \in J} (g_k, A g_k)$$

Articles

C. Jankowski, D. Markiewicz, and R. T. Powers, *E_0-semigroups and q-purity: boundary weight maps of range rank one and two*, J. Funct. Anal. **262** (2012), no. 7, 3006–3061.

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Thank you!