An operator theory approach to orthogonal polynomials in several variables

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OTOA 2018, Indian Statistical Institute, Bangalore
December 13-19, 2018

Joint work with J. Stochel and F.H. Szafraniec

Notations

$\mathbb{N} = \{0, 1, 2, \ldots\}$

Measure on $\mathbb{R}^N$ = Borel measure on $\mathbb{R}^N$ (with all moments finite, i.e. $\int_{\mathbb{R}^N} \|x\|^n \mathrm{d}\mu(x) < \infty$ for all $n \geqslant 1$).

$\mathcal{P}_N$ – space of all complex polynomials in N variables.

$\mathcal{P}_N^k$ – space of all complex polynomials in N variables of degree at most k.

A measure μ on $\mathbb{R}^N$ *orthonormalizes* a sequence of real polynomials $\{p_k\}_{k=0}^{\infty}$ if $\int_{\mathbb{R}^N} p_k p_l \mathrm{d}\mu = \delta_{kl}$.

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Theorem (Favard)

If $\{p_k\}_{k=0}^{\infty}$ is a sequence of real polynomials in one variable such that $p_0 = 1$ and $\deg p_k = k$ for all $k \in \mathbb{N}$, then the following two conditions are equivalent:

(i) *there exists a measure μ on $\mathbb{R}$ which orthonormalizes $\{p_k\}_{k=0}^{\infty}$,*

(ii) *for every $k \in \mathbb{N}$, there exist $a_k \in \mathbb{R}$ and $b_k \in \mathbb{R}$ such that*

$$Xp_k = a_k p_{k+1} + b_k p_k + a_{k-1} p_{k-1}, \quad \text{where } a_{-1} \stackrel{\text{def}}{=} 1 \text{ and } p_{-1} \stackrel{\text{def}}{=} 0.$$

If (i) *holds, then* $\operatorname{supp} \mu$ *is infinite.*

The condition (2) is called the three term recurrence relation.

How can we make a several variable version?

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Background

The succesful attempts were made by Kowalski (1982) and Xu (1994), but they formulated their versions for full polynomial bases of $\mathcal{P}_N$.

This clearly excludes some "decent" measures, e.g. the Lebesgue measure on the unit circle in $\mathbb{R}^2$.

Reason: every polynomial is orthogonal to $x^2 + y^2 - 1$ with respect to this measure.

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Notations

Let $V \subset \mathcal{P}_N$ be an ideal and $\Pi_V : \mathcal{P}_N \to \mathcal{P}_N/V$ be the canonical embedding. An ideal V is called a $*$-ideal if $\overline{p} \in V$ whenever $p \in V$ ($\overline{p}$ – the coefficients are complex conjugates).

For simplicity we consider only the case $\dim \mathcal{P}_N/V = \infty$, though the theory also works if $\dim \mathcal{P}_N/V$ is finite.

Let $d_V(k) = \dim \Pi_V(\mathcal{P}_N^k) - \dim \Pi_V(\mathcal{P}_N^{k-1})$, $k \geqslant 1$ and $d_V(0) = 1$.

A sequence $\{Q_k\}_{k=0}^{\infty}$ is called a *rigid V-basis* of $\mathcal{P}_N$ if every Q_k is a column polynomial of size $d_V(k)$, i.e.

$$Q_k = \begin{bmatrix} q_1^{(k)} \\ \vdots \\ q_{d_V(k)}^{(k)} \end{bmatrix}, \quad q_j^{(k)} \in \mathcal{P}_N,$$

all polynomials in Q_k are of degree k and the set

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If P and Q are column polynomials, then $P \overset{V}{=} Q$ means that the columns are of the same size and entries of $P - Q$ are in V.

If $L : \mathcal{P}_N \to \mathbb{C}$ is a linear functional, then we write

$$L([p_{k,l}]_{k=0}^{m}{}_{l=0}^{n}) = [L(p_{k,l})]_{k=0}^{m}{}_{l=0}^{n},$$

where $p_{k,l} \in \mathcal{P}_N$.

This way we can make sense of $L(PQ^{\intercal})$, where P and Q are column polynomials (not necessarily of the same size!).

Finally we say that L orthonormalizes the rigid V-basis $\{Q_k\}_{k=0}^{\infty}$, if $L(Q_k Q_l^{\intercal}) = 0$ when $k \neq l$, and $L(Q_k Q_k^{\intercal}) = I$, $k \geqslant 1$.

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Generalized Favard's Theorem

A linear functional $L : \mathcal{P}_N \to \mathbb{C}$ is called positive definite if $L(p\overline{p}) \geqslant 0$ for all $p \in \mathcal{P}_N$.

Theorem

Let $V \subset \mathcal{P}_N$ be an $*$-ideal and $\{Q_k\}_{k=0}^{\infty}$ be a rigid V-basis of real polynomials with $Q_0 = 1$. Then the following conditions are equivalent:

(A) *there exists positive definite $L : \mathcal{P}_N \to \mathbb{C}$ which orthonormalizes $\{Q_k\}_{k=0}^{\infty}$ and such that $V \subset \ker L$;*

(B) *there exists systems of scalar matrices $\{A_{k,j}\}_{k=0}^{\infty}{}_{j=1}^{N}$ and $\{B_{k,j}\}_{k=0}^{\infty}{}_{j=1}^{N}$ of appropriate sizes such that*

$$X_j Q_k \overset{V}{=} A_{k,j} Q_{k+1} + B_{k,j} Q_k + A_{k-1,j}^{\mathsf{T}} Q_{k-1},$$

for all $j = 1, \ldots, N$, $k \in \mathbb{N}$ $(A_{-1,j} = 1,\ Q_{-1} = 0)$.

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Christoffel-Darboux kernel

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Let $\{Q_k\}_{k=0}^{\infty}$ *be a rigid* V*-basis of real polynomials with* $Q_0 = 1$ *satisfying the three term recurrence relation* (B). *Fix* $j \in \{1, \dots, N\}$. *Then*

$$K_n(x, y) \stackrel{\text{def}}{=} \sum_{k=0}^{n} Q_k^{\intercal}(x) Q_k(y)$$

is equivalent to

$$\tilde{K}_{n,j}(x, y) = \frac{[A_{n,j} Q_{n+1}(x)]^{\intercal} Q_n(y) - Q_n(x)^{\intercal} [A_{n,j} Q_{n+1}(y)]}{x_j - y_j}$$

modulo the ideal $V \otimes \mathcal{P}_N + \mathcal{P}_N \otimes V$.

The converse is also true, i.e. if K_n and $\tilde{K}_{n,j}$ are defined as above and are equal up to the ideal, then (B) holds.

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Remarks

The proof of these theorems can be carried out with the help of pure linear algebra.

Once (B) is assumed, L may be defined via $L(Q_0) = 1$, $L(Q_k) = 0$, $k \in \mathbb{N}$, and $L|_V = 0$. This the only possible choice. Moreover, V is equal to $\mathcal{V}_L$, the largest $*$-ideal contained in $\ker L$

Contrary to the original Favard's theorem we have a functional L instead of a measure. This is due to the fact that every positive definite functional $L : \mathcal{P}_1 \to \mathbb{C}$ is a moment functional, i.e. there exists a measure on $\mathbb{R}$ such that

$$L(p) = \int_{\mathbb{R}} p \mathrm{d}\mu \text{ for all } p \in \mathcal{P}_1.$$

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Assume (B) holds and L is a moment functional given by measure μ on $\mathbb{R}^N$. Then $\mathcal{V}_L$ is the set ideal of $\operatorname{supp}\mu$, i.e.

$$\mathcal{I}(\operatorname{supp}\mu) \stackrel{\text{def}}{=} \{q \in \mathcal{P}_N \colon q|_{\operatorname{supp}\mu} = 0\}.$$

Reason:
$q \in \mathcal{V}_L \Rightarrow \int_{\mathbb{R}^N} |q|^2 \mathrm{d}\mu = 0 \Rightarrow q|_{\operatorname{supp}\mu} = 0 \Rightarrow q \in \mathcal{I}(\operatorname{supp}\mu)$.
(The reverse is even more obvious.)

In turn, $\mathcal{I}(\operatorname{supp}\mu) = \mathcal{I}(\Delta)$, where Δ is Zariski closure of $\operatorname{supp}\mu$, i.e. the smallest real algebraic set containing $\operatorname{supp}\mu$.

This means that we may find polynomial p such that $\mathcal{V}_L = \mathcal{I}(\mathcal{Z}_p)$, where $\mathcal{Z}_p = \{x \in \mathbb{R}^N \colon p(x) = 0\}$.

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Fix $p \in \mathcal{P}_N$ and assume that $V = \mathcal{I}(\mathcal{Z}_p)$ and (B) holds.

Assume that p satisfies the following condition:

(A^0) for every inner product space $\mathcal{D}$ and every N-tuple $S = (S_1, \ldots, S_N)$ of commuting linear operators $\mathcal{D} \to \mathcal{D}$ such that $\langle S_j f, g \rangle = \langle f, S_j g \rangle$ ($j = 1, \ldots, N$, $f, g \in \mathcal{D}$) and $p(S) = 0$ there exists a Hilbert space $\mathcal{K} \supset \mathcal{D}$ and N-tuple $T = (T_1, \ldots, T_N)$ of spectrally commuting selfadjoint operators in $\mathcal{K}$ such that T_j is an extension of S_j.

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Examples

$\boxed{1}$ $p(x,y) = x^2 + y^2 - 1$ – the case of the unit circle in $\mathbb{R}^2$. If we take a pair $S = (S_1, S_2)$ of commuting symmetric operators on $\mathcal{D}$, then $p(S) = 0$ means that $S_1^2 + S_2^2 = I$, so $\|S_1 f\|^2 + \|S_2 f\|^2 = \|f\|^2$, $f \in \mathcal{D}$, which implies that S_1 and S_2 are bounded, and can be extended to $\mathcal{K}$, the completion of $\mathcal{D}$. Commutativity is preserved, so (A^0) holds.

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$\boxed{3}$ $p(x,y) = 0$ – the case in which $\mathcal{V}_L = \{0\}$. It is not of type (A^0), however $p(x) = 0$ is! What is more, there exists a positive definite linear functional on $\mathcal{P}_2$ which is not a moment functional.

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Property (A^0)

Theorem (Schmüdgen's theorem adapted to our settings)

If $p \in \mathcal{P}_N$ is such that $\mathcal{Z}_p$ is compact, then p satisfies (A^0).

Property (A^0) is preserved under

- freezing variables:
  $p(x_1, \ldots, x_N) \in (A^0) \Rightarrow p(x_1, \ldots, x_k, \lambda_{k+1}, \ldots, \lambda_N) \in (A^0)$;
- substitution of a polynomial automorphism:
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Theorem

Assume that the condition (B) *of the generalized Favard's theorem is satisfied and there is a sequence* $\{c_n\}_{n=0}^{\infty}$ *of positive numbers such that* $\sum_{n=0}^{\infty} \frac{1}{\sqrt{c_n}} = \infty$, *and*

$$c_n \geqslant \max\Big\{\Big\|\sum_{j=1}^{N} A_{n-1,j}A_{n,j}\Big\|, \Big\|\sum_{j=1}^{N} A_{n,j}A_{n+1,j}\Big\|, \Big\|\sum_{j=1}^{N}\left(B_{n,j}A_{n,j} + A_{n,j}B_{n+1,j}\right)\Big\|\Big\}.$$

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This is a consequence of the Nelson criterion applied to the N-tuple operators $(S_1, \ldots, S_N)$ assuring their spectral commutativity by means of $S_1^2 + \ldots + S_N^2$.

Operator S_j is an operator of multiplication by X_j, its matrix representation in the basis $\{Q_k\}_{k=0}^{\infty}$ is given by the system of matrices $\{A_{k,j}\}_{k=0}^{\infty}{}_{j=1}^{N}$ and $\{B_{k,j}\}_{k=0}^{\infty}{}_{j=1}^{N}$, and the expressions appearing above under "max" are related to matrix representation of $S_1^2 + \ldots + S_N^2$.

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Orthogonal polynomials on the sphere

We now focus on the normalized Lebesgue measure on the unit sphere S_N in $\mathbb{R}^{N+1}$. We may consider orthonormal basis of $\mathcal{P}_N$ with respect to the following two weights defined on the unit ball $B_N \subset \mathbb{R}^N$:

$$W_1(x) = \frac{1}{\sqrt{1 - \|x\|^2}} \text{ and } W_2(x) = \sqrt{1 - \|x\|^2}$$

They can be written in the form of rigid bases for $\mathcal{P}_N$ (with ideal understood to be the zero ideal), denote them by $\{P_k\}_{k=1}^{\infty}$ and $\{Q_k\}_{k=1}^{\infty}$, respectively, interpreted as dependent on $\hat{x} = (x_1, \ldots, x_N)$.

Then the sequence defined by

$$Y_0 = 1 \text{ and } Y_n(x) = \begin{bmatrix} P_k(\hat{x}) \\ x_{N+1} Q_{k-1}(\hat{x}) \end{bmatrix}, \quad \text{where } x = (\hat{x}, x_{N+1}).$$

forms a rigid V-basis of $\mathcal{P}_{N+1}$, where $V = \mathcal{I}(S_N)$.

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It can be shown that each polynomial in the column Y_k is equal to a homogeneous polynomial modulo $V = \mathcal{I}(S_N)$. Reason: they are sums of monomials of degree always odd or always even. Example for $N = 1$:

$$x_1 + x_2^3 \stackrel{V}{=} (x_1^2 + x_2^2)x_1 + x_2^3.$$

Since the lengths of columns Y_k in the rigid V-basis depends only on the ideal V, we see that the structure of any other rigid V-basis (i.e. the lengths of consecuting columns) will be the same if the Lebesgue measure on S_N is replaced by any other measure with "sufficiently big" support.

In the case $N = 1$ the basis $\{Y_k\}_{k=0}^{\infty}$ can be written in terms of Chebyshev polynomials of the first and the second kind.

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Since the lengths of columns Y_k in the rigid V-basis depends only on the ideal V, we see that the structure of any other rigid V-basis (i.e. the lengths of consecuting columns) will be the same if the Lebesgue measure on S_N is replaced by any other measure with "sufficiently big" support.

In the case $N = 1$ the basis $\{Y_k\}_{k=0}^{\infty}$ can be written in terms of Chebyshev polynomials of the first and the second kind.

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Thank You