On some extension of pairs of commuting isometries.

Zbigniew Burdak

Department of Applied Mathematics, Kraków

18 December 2018, OTOA Bangalore

notation

- $\mathcal{B}(H)$ - the algebra of bounded linear operators on a separable, complex Hilbert space H,
- $\mathrm{Lat}(S)$ - the lattice of S - invariant subspaces, $S \in \mathcal{B}(H)$,
- $L^2_{\mathcal{H}}(\mathbb{T})$ - the space of square integrable, $\mathcal{H}$ valued functions, where $\mathcal{H}$ is a complex Hilbert space,
- $H^2_{\mathcal{H}}(\mathbb{T})$ - Hardy space of $\mathcal{H}$ valued functions,
- $M_z \in \mathcal{B}(L^2_{\mathcal{H}}(\mathbb{T}))$, $T_z \in \mathcal{B}(H^2_{\mathcal{H}}(\mathbb{T}))$ operators of multiplication by the independent variable $''z''$,

Beurling-Lax-Halmos theorem

$T_z \in \mathcal{B}(H^2_{\mathcal{H}}(\mathbb{T}))$ is a model of a unilateral shift of multiplicity $\dim \mathcal{H}$.

$\phi : \mathbb{T} \mapsto \mathcal{B}(\mathcal{H})$ is an inner function iff $\phi(z)$ are partial isometries with the same initial space for almost every z.

$M_\phi \in \mathcal{B}(H^2_{\mathcal{H}}(\mathbb{T}))$ where $M_\phi f : z \mapsto \phi(z)f(z)$.

Theorem (Beurling-Lax-Halmos, 1961)

Invariant subspaces of $T_z \in H^2_{\mathcal{H}}(\mathbb{T})$ are precisely subspaces of the form

$$M_\phi H^2_{\mathcal{H}}(\mathbb{T})$$

where ϕ is an inner function.

Beurling-Lax-Halmos theorem

$T_z \in \mathcal{B}(H^2_{\mathcal{H}}(\mathbb{T}))$ is a model of a unilateral shift of multiplicity $\dim \mathcal{H}$.

$\phi : \mathbb{T} \mapsto \mathcal{B}(\mathcal{H})$ is an inner function iff $\phi(z)$ are partial isometries with the same initial space for almost every z.

$M_\phi \in \mathcal{B}(H^2_{\mathcal{H}}(\mathbb{T}))$ where $M_\phi f : z \mapsto \phi(z) f(z)$.

Theorem (Beurling-Lax-Halmos, 1961)

Invariant subspaces of $T_z \in H^2_{\mathcal{H}}(\mathbb{T})$ are precisely subspaces of the form

$$M_\phi H^2_{\mathcal{H}}(\mathbb{T})$$

where ϕ is an inner function.

Beurling-Lax-Halmos theorem

$T_z \in \mathcal{B}(H^2_{\mathcal{H}}(\mathbb{T}))$ is a model of a unilateral shift of multiplicity $\dim \mathcal{H}$.

$\phi : \mathbb{T} \mapsto \mathcal{B}(\mathcal{H})$ is an inner function iff $\phi(z)$ are partial isometries with the same initial space for almost every z.

$M_\phi \in \mathcal{B}(H^2_{\mathcal{H}}(\mathbb{T}))$ where $M_\phi f : z \mapsto \phi(z) f(z)$.

Theorem (Beurling-Lax-Halmos, 1961)

Invariant subspaces of $T_z \in H^2_{\mathcal{H}}(\mathbb{T})$ are precisely subspaces of the form

$$M_\phi H^2_{\mathcal{H}}(\mathbb{T})$$

where ϕ is an inner function.

commuting unilateral shifts

Let $(S_1, S_2) \in \mathcal{B}(H)$ be a pair of commuting unilateral shifts.

$\mathrm{Lat}(S_1, S_2)$ - the lattice of joint invariant subspaces.

$\mathrm{Lat}(S_1, S_2) = \mathrm{Lat}(S_1) \cap \mathrm{Lat}(S_2)$

$S_i \simeq T_z \in \mathcal{B}(H^2_{\mathcal{H}_i}(\mathbb{T}))$ where $\mathcal{H}_i \simeq \ker S_i^*$ for $i = 1, 2$.

$$\mathrm{Lat}(S_1) \cap \mathrm{Lat}(S_2)$$

is difficult to calculate.

commuting unilateral shifts

Let $(S_1, S_2) \in \mathcal{B}(H)$ be a pair of commuting unilateral shifts.

$\mathrm{Lat}(S_1, S_2)$ - the lattice of joint invariant subspaces.

$\mathrm{Lat}(S_1, S_2) = \mathrm{Lat}(S_1) \cap \mathrm{Lat}(S_2)$

$S_i \simeq T_z \in \mathcal{B}(H^2_{\mathcal{H}_i}(\mathbb{T}))$ where $\mathcal{H}_i \simeq \ker S_i^*$ for $i = 1, 2$.

$$\mathrm{Lat}(S_1) \cap \mathrm{Lat}(S_2)$$

is difficult to calculate.

commuting unilateral shifts

Let $(S_1, S_2) \in \mathcal{B}(H)$ be a pair of commuting unilateral shifts.

$\mathrm{Lat}(S_1, S_2)$ - the lattice of joint invariant subspaces.

$\mathrm{Lat}(S_1, S_2) = \mathrm{Lat}(S_1) \cap \mathrm{Lat}(S_2)$

$S_i \simeq T_z \in \mathcal{B}(H^2_{\mathcal{H}_i}(\mathbb{T}))$ where $\mathcal{H}_i \simeq \ker S_i^*$ for $i = 1, 2$.

$$\mathrm{Lat}(S_1) \cap \mathrm{Lat}(S_2)$$

is difficult to calculate.

commuting unilateral shifts

Let $(S_1, S_2) \in \mathcal{B}(H)$ be a pair of commuting unilateral shifts.

$\mathrm{Lat}(S_1, S_2)$ - the lattice of joint invariant subspaces.

$\mathrm{Lat}(S_1, S_2) = \mathrm{Lat}(S_1) \cap \mathrm{Lat}(S_2)$

$S_i \simeq T_z \in \mathcal{B}(H^2_{\mathcal{H}_i}(\mathbb{T}))$ where $\mathcal{H}_i \simeq \ker S_i^*$ for $i = 1, 2$.

$$\mathrm{Lat}(S_1) \cap \mathrm{Lat}(S_2)$$

is difficult to calculate.

description via extension

$(V_1, V_2) \in \mathcal{B}(H)$ - a pair of commuting isometries,

$(\tilde{V}_1, \tilde{V}_2) \in \mathcal{B}(\tilde{H})$ - an isometric extension of (V_1, V_2).

Then:

$H \in \mathrm{Lat}(\tilde{V}_1, \tilde{V}_2)$,

$$\mathrm{Lat}(V_1, V_2) = \{\mathcal{M} \cap H : \mathcal{M} \in \mathrm{Lat}(\tilde{V}_1, \tilde{V}_2)\}$$

description via extension

$(V_1, V_2) \in \mathcal{B}(H)$ - a pair of commuting isometries,

$(\tilde{V}_1, \tilde{V}_2) \in \mathcal{B}(\tilde{H})$ - an isometric extension of (V_1, V_2).

Then:

$H \in \mathrm{Lat}(\tilde{V}_1, \tilde{V}_2)$,

$$\mathrm{Lat}(V_1, V_2) = \{\mathcal{M} \cap H : \mathcal{M} \in \mathrm{Lat}(\tilde{V}_1, \tilde{V}_2)\}$$

Aim:

1. for a given relatively prime, positive integers m, n extend an arbitrary pair of isometries to a pair
$$(U^k V^m, U^l V^n),$$
where:
U is a unitary operator commuting with an isometry V, and $km - ln = 1$,
2. describe a model of the pair
$$(U^k V^m, U^l V^n),$$
3. describe
$$\mathrm{Lat}(U^k V^m, U^l V^n).$$
(partial results)

the extension

Aim:

1. for a given relatively prime, positive integers m, n extend an arbitrary pair of isometries to a pair

$$(U^k V^m, U^l V^n),$$

   where:
   U is a unitary operator commuting with an isometry V, and $km - ln = 1$,
2. describe a model of the pair

$$(U^k V^m, U^l V^n),$$

3. describe

$$\mathrm{Lat}(U^k V^m, U^l V^n).$$

   (partial results)

the extension

Aim:

1. for a given relatively prime, positive integers m, n extend an arbitrary pair of isometries to a pair

$$(U^k V^m, U^l V^n),$$

   where:
   U is a unitary operator commuting with an isometry V, and $km - ln = 1$,
2. describe a model of the pair

$$(U^k V^m, U^l V^n),$$

3. describe

$$\mathrm{Lat}(U^k V^m, U^l V^n).$$

   (partial results)

extension

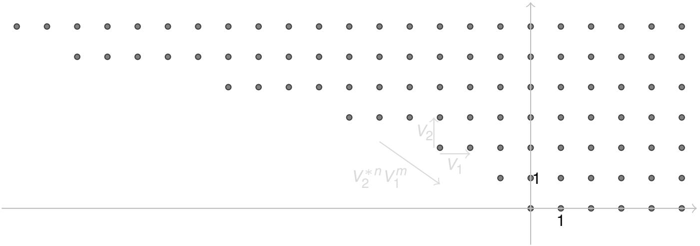
Proposition

For any pair of commuting isometries $(V_1, V_2) \in \mathcal{B}(H)$ *and positive integers* m, n*, there is an extension to a commuting pair of isometries* $(\widehat{V}_1, \widehat{V}_2)$ *on a Hilbert space* $\widehat{H}$ *where*

$$\widehat{V}_2^{*n}\widehat{V}_1^{m}$$

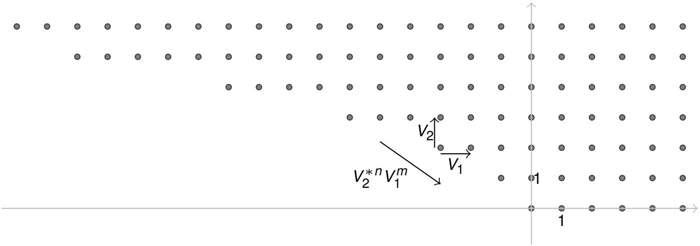
is a unitary operator commuting with $\widehat{V}_1, \widehat{V}_2$*. Moreover, the extension may be chosen to be minimal.*

$\mathcal{M} = \{f \in L^2(\mathbb{T}^2) : \hat{f}_{i,j} = 0 \text{ for } (i,j) \in \mathbb{Z}^2 \setminus Z\} \subset L^2(\mathbb{T}^2)$ where Z is as in the picture



For $V_1 = M_{z_1}|_{\mathcal{M}}, V_2 = M_{z_2}|_{\mathcal{M}}$ a minimal extension $(\widehat{V}_1, \widehat{V}_2)$ such that $\widehat{V}_2^{*n} \widehat{V}_1^m$ is unitary is M_{z_1}, M_{z_2} for any m, n.

$\mathcal{M} = \{f \in L^2(\mathbb{T}^2) : \hat{f}_{i,j} = 0 \text{ for } (i,j) \in \mathbb{Z}^2 \setminus Z\} \subset L^2(\mathbb{T}^2)$ where Z is as in the picture



For $V_1 = M_{z_1}|_{\mathcal{M}}, V_2 = M_{z_2}|_{\mathcal{M}}$ a minimal extension $(\widehat{V}_1, \widehat{V}_2)$ such that $\widehat{V}_2^{*n} \widehat{V}_1^m$ is unitary is M_{z_1}, M_{z_2} for any m, n.

extension

Theorem

A pair of commuting isometries (V_1, V_2) *on a Hilbert space* H *such that for some relatively prime, positive integers* m, n *the operator*

$$V_2^{*n} V_1^m \text{ is unitary}$$

may be extended to a pair

$$(\tilde{U}^k \tilde{V}^n, \tilde{U}^l \tilde{V}^m)$$

where:

- $\tilde{U}$ *is a unitary operator commuting with an isometry* $\tilde{V}$*,*
- $H \in \mathrm{Lat}(\tilde{V}^m, \tilde{V}^n)$ *and*
- (k, l) *are unique integers such that* $0 < k < n, 0 \leq l < m$ *and* $km - ln = 1$*.*

Moreover, the extension may be chosen to be minimal, and for a minimal extension if V_1, V_2 *are unilateral shifts, then* $\tilde{V}$ *is a unilateral shift.*

Theorem

Any pair of commuting isometries (V_1, V_2), *for any relatively prime, positive integers* m, n *may be extended to a pair*

$$(\widehat{U}^k \widehat{V}^n, \widehat{U}^l \widehat{V}^m)$$

where $\widehat{U}$ *is a unitary operator commuting with an isometry* $\widehat{V}$ *and* (k, l) *are unique integers such that* $0 < k < n, 0 \leq l < m$ *and* $km - ln = 1$. *Moreover, the extension may be chosen minimal.*

Let m, n be relatively prime, positive integers and $km - ln = 1$. Any pair of the form

$$(U^k V^n, U^l V^m)$$

where U is a unitary operator commuting with an isometry V is unitarily equivalent to:

$$(U_1 \oplus (T_z^n \otimes \mathcal{U}^k), U_2 \oplus (T_z^m \otimes \mathcal{U}^l))$$

on the Hilbert space $H_u \oplus (H^2(\mathbb{T}) \otimes \mathcal{H})$ for the respective unitary operators $U_1, U_2 \in \mathcal{B}(H_u)$, $\mathcal{U} \in \mathcal{B}(\mathcal{H})$.

$\mathrm{Lat}(V^m, V^n)$

Theorem

Let $T_z \in \mathcal{B}(H^2_{\mathcal{H}}(\mathbb{T}))$ and m, n be relatively prime, positive integers. The subspaces jointly invariant under (T_z^m, T_z^n) are precisely those of the form

$$M_\phi\left(H_0 \oplus (I - P)H^2_{\mathcal{H}_0}(\mathbb{T})\right)$$

where

- $P \in \mathcal{B}(H^2_{\mathcal{H}}(\mathbb{T}))$ *is an orthogonal projection on the space of polynomials of degree at most* $mn - m - n$,
- ϕ *is an inner function with initial space* $\mathcal{H}_0$ *and*
- $H_0 \subset PH^2_{\mathcal{H}_0}(\mathbb{T})$ *invariant under* PT_z^m, PT_z^n.

- $PT_z^3 = PT_z^2 = 0$ (the case $m = 3, n = 2$),
- if $\dim \mathcal{H}_0 < \infty$ then $\dim PH^2_{\mathcal{H}_0}(\mathbb{T}) < \infty$

Lat(UV, V) $(m = n = 1)$

$$H^2_{\mathcal{H}}(\mathbb{T}) \simeq H^2(\mathbb{T}) \otimes \mathcal{H}$$

$$V \simeq T_z \otimes I,$$

$$U \simeq I \otimes \mathcal{U}.$$

Theorem

The subspaces jointly invariant under $(T_z \otimes I, T_z \otimes \mathcal{U})$ *are precisely those of the form* $M_\phi(H^2(\mathbb{T}) \otimes \mathcal{H})$ *where* ϕ *is an inner function satisfying*

$$M_\phi(H^2(\mathbb{T}) \otimes \mathcal{H}) = WM_\psi(H^2(\mathbb{T}) \otimes \mathcal{H})$$

with some other inner function ψ *and* $W = \sum_{i \geq 0} P_{\mathbb{C}z^i} \otimes \mathcal{U}^i$.

$H^2(\mathbb{T}^2)$

$$T_{z_1}, T_{z_2} \in \mathcal{B}(H^2(\mathbb{T}^2)) \text{ and } M_{z_1}, M_{z_2} \in \mathcal{B}(L^2(\mathbb{T}^2))$$

(T_{z_1}, T_{z_2}) extends to a pair of unilateral shifts $(\tilde{T}_{z_1}, \tilde{T}_{z_2})$ such that

$$\tilde{T}^*_{z_1}\tilde{T}_{z_2} \text{ is unitary,}$$

where

$$\tilde{T}_{z_\alpha} = M_{z_\alpha}|_{\mathcal{M}} \text{ for } \alpha = 1, 2$$

and $\mathcal{M} := \{f \in L^2(\mathbb{T}^2) : \hat{f}_{i,j} = 0 \text{ for } j < -i\}$.

$$\mathrm{Lat}(\tilde{T}_{z_1}, \tilde{T}_{z_2}) = \mathrm{Lat}(\tilde{T}_{z_1}) \cap W\mathrm{Lat}(\tilde{T}_{z_1})$$

where $W \in \mathcal{B}(\mathcal{M})$ is defined by

$$W z_1^i z_2^j = z_1^{2i+j} z_2^{-i}.$$

$$T_{z_1}, T_{z_2} \in \mathcal{B}(H^2(\mathbb{T}^2)) \text{ and } M_{z_1}, M_{z_2} \in \mathcal{B}(L^2(\mathbb{T}^2))$$

(T_{z_1}, T_{z_2}) extends to a pair of unilateral shifts $(\tilde{T}_{z_1}, \tilde{T}_{z_2})$ such that

$$\tilde{T}_{z_1}^* \tilde{T}_{z_2} \text{ is unitary,}$$

where

$$\tilde{T}_{z_\alpha} = M_{z_\alpha}|_{\mathcal{M}} \text{ for } \alpha = 1, 2$$

and $\mathcal{M} := \{f \in L^2(\mathbb{T}^2) : \hat{f}_{i,j} = 0 \text{ for } j < -i\}$.

$$\mathrm{Lat}(\tilde{T}_{z_1}, \tilde{T}_{z_2}) = \mathrm{Lat}(\tilde{T}_{z_1}) \cap W\mathrm{Lat}(\tilde{T}_{z_1})$$

where $W \in \mathcal{B}(\mathcal{M})$ is defined by

$$W z_1^i z_2^j = z_1^{2i+j} z_2^{-i}.$$

$H^2(\mathbb{T}^2)$

$$T_{z_1}, T_{z_2} \in \mathcal{B}(H^2(\mathbb{T}^2)) \text{ and } M_{z_1}, M_{z_2} \in \mathcal{B}(L^2(\mathbb{T}^2))$$

(T_{z_1}, T_{z_2}) extends to a pair of unilateral shifts $(\tilde{T}_{z_1}, \tilde{T}_{z_2})$ such that

$$\tilde{T}^*_{z_1} \tilde{T}_{z_2} \text{ is unitary,}$$

where

$$\tilde{T}_{z_\alpha} = M_{z_\alpha}|_{\mathcal{M}} \text{ for } \alpha = 1, 2$$

and $\mathcal{M} := \{f \in L^2(\mathbb{T}^2) : \hat{f}_{i,j} = 0 \text{ for } j < -i\}$.

$$\mathrm{Lat}(\tilde{T}_{z_1}, \tilde{T}_{z_2}) = \mathrm{Lat}(\tilde{T}_{z_1}) \cap W\mathrm{Lat}(\tilde{T}_{z_1})$$

where $W \in \mathcal{B}(\mathcal{M})$ is defined by

$$W z_1^i z_2^j = z_1^{2i+j} z_2^{-i}.$$

finite multiplicity

Consider (V_1, V_2) a pair of unilateral shifts such that $U := V_2^{*n} V_1^m$ is unitary. Then

$$V_1^m \simeq T_z \otimes I, \ V_2^n \simeq T_z \otimes \mathcal{U}, \ U = I \otimes \mathcal{U}.$$

If V_1, V_2 are of finite multiplicity then $\mathcal{U}$ is a unitary operator on a finite dimensional space.
Eigenvalues/eigenspaces of $\mathcal{U}$ corresponds to those of $U := I \otimes \mathcal{U}$ which commutes with V_1, V_2.

finite multiplicity

Remark

Let a pair of commuting unilateral shifts (V_1, V_2) on H satisfy

$$V_2^{*n} V_1^m = \lambda I$$

for relatively prime, positive integers m, n and a complex number λ. Then there is a unilateral shift $\tilde{V} \in \mathcal{B}(\tilde{H})$ such that $H \subset \tilde{H}$ and

$$\mathrm{Lat}(V_1, V_2) = \{H \cap \mathcal{N} : \mathcal{N} \in \mathrm{Lat}(\tilde{V}^m, \tilde{V}^m)\}$$

where $\mathrm{Lat}(\tilde{V}^n, \tilde{V}^m)$ is described.

Thank You !