Isometric dilation and von Neumann inequality for operator tuples

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(joint work with S. Barik, K. Haria and J. Sarkar)

Definition

Let T be a contraction on H. A unitary U on $K \supseteq H$ is a dilation of T if (?) $T = P_H U|_H$, i.e.

$$U = \begin{bmatrix} T & * \\ * & * \end{bmatrix}$$

with respect to the decomposition $K = H \oplus H^{\perp}$.

Introduction

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Let T be a contraction on H. A unitary U on $K \supseteq H$ is a dilation of T if $T^n = P_H U^n|_H$ for all $n \in \mathbb{N}$, i.e.

$$U^n = \begin{bmatrix} T^n & * \\ * & * \end{bmatrix}$$

with respect to the decomposition $K = H \oplus H^\perp$. In this case, $p(T) = P_H p(U)|_H$ for any polynomial $p \in \mathbb{C}[z]$.

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Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting contractions on H. An n-tuple of commuting unitary $U = (U_1, \ldots, U_n)$ on $K \supseteq H$ is a dilation of T if $p(T) = P_H p(U)|_H$ for any polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$, i.e.

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Nagy-Foias and Ando dilation

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Let (T_1, T_2) be a pair of commuting contractions on $\mathcal{H}$. *Then (T_1, T_2) dilates to a pair of commuting unitaries (U_1, U_2).*

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- Neither dilation nor the von Neumann inequality holds for n-tuples of commuting contractions with $n > 2$

Hardy space and Szegö positivity

- $\mathcal{E}$-valued Hardy space over the unit polydisc $\mathbb{D}^n$:

$$H^2_{\mathcal{E}}(\mathbb{D}^n) := \{\sum_{k\in\mathbb{Z}^n} a_k z^k : a_k \in \mathcal{E}, \sum_{k\in\mathbb{Z}^n} \|a_k\|^2 < \infty\}.$$

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- $H^2_{\mathcal{E}}(\mathbb{D}^n)$ is a reproducing kernel Hilbert space with kernel $\mathbb{S}_n(z, w) I_{\mathcal{E}}$ where $\mathbb{S}_n$ is the Szegö kernel defined by
$$\mathbb{S}_n(z, w) = \prod_{i=1}^{n} (1 - z_i \bar{w}_i)^{-1} \quad (z, w \in \mathbb{D}^n).$$

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- An n-tuple of commuting contractions $T = (T_1, \dots, T_n)$ satisfies Szegö positivity if

$$\mathbb{S}_n^{-1}(T, T^*) = \sum_{F \subset \{1, \dots, n\}} (-1)^{|F|} T_F T_F^* \geqslant 0$$

where for $F = \{F_1, \dots, F_k\} \subset \{1, \dots, n\}$, $T_F := T_{F_1} \cdots T_{F_k}$.

Dilation under Szegö positivity

- If $T = (T_1, \ldots, T_n)$ satisfy Szegö positivity then $D_T := \mathbb{S}_n^{-1}(T, T^*)^{1/2}$ is the defect operator and $\mathcal{D}_T := \overline{\mathrm{Ran}} D_T$ is the defect space of T.

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Theorem (Curto & Vasilescu, 1993)

Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting pure contractions which satisfy Szegö positivity. Then T dilates to $(M_{z_1}, \ldots, M_{z_n})$ on $H^2_{\mathcal{D}_T}(\mathbb{D}^n)$.

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Remark: The hypothesis of the above theorem is also necessary for an n-tuple of commuting contractions T to be dilated to $(M_{z_1}, \ldots, M_{z_n})$ on some $H^2_{\mathcal{E}}(\mathbb{D}^n)$.

Class of operator tuples

- For an n-tuple $T = (T_1, \ldots, T_n)$ and $1 \leq r \leq n$, $\hat{T}_r = (T_1, \ldots, T_{r-1}, T_{r+1}, \ldots, T_n)$.

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- $\mathcal{P}_{p,q} = \{(T_1, \dots, T_n) : \hat{T}_p, \hat{T}_q$ satisfy Szegö positivity and $\|T_i\| < 1, i = 1, \dots, n\}$.

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Example: Let $T_i = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}$ for all $i = 1, \ldots, n$ and let $\frac{1}{\sqrt{n}} < r \le \frac{1}{\sqrt{n-1}}$. Then for any $1 \le i \le n$,

$$\mathbb{S}_{n-1}^{-1}(\hat{T}_i, \hat{T}_i^*) = \begin{bmatrix} 1 - r^2(n-1) & 0 \\ 0 & 1 \end{bmatrix} \ge 0$$

and

$$\mathbb{S}_n^{-1}(T, T^*) = \begin{bmatrix} 1 - r^2 n & 0 \\ 0 & 1 \end{bmatrix} \not\ge 0.$$

Thus $T \in \mathcal{P}_{p,q}$ for any $1 \le p < q \le n$. But T does not satisfy Szegö positivity.

Theorem (Vinnikov et all, 2009)

Let T be an n-tuple of commuting contractions. If $T \in \mathcal{P}_{p,q}$ for some $1 \leq p < q \leq n$, then T can be dilated to a tuple of commuting unitaries. Therefore, T satisfies von Neumann inequality on $\mathbb{D}^n$.

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- $\mathcal{T}_{p,q} = \{(T_1, \ldots, T_n) : \hat{T}_p, \hat{T}_q$ satisfy Szegö positivity and $\hat{T}_p$ is pure$\}$.

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- $\mathcal{T}_{p,q} = \{(T_1, \dots, T_n) : \hat{T}_p, \hat{T}_q$ satisfy Szegö positivity and $\hat{T}_p$ is pure$\}$.
- $\mathcal{P}_{p,q} \subseteq \mathcal{T}_{p,q}$.
- For $T \in \mathcal{T}_{p,q}$, we say that T is of finite rank if $\dim \mathcal{D}_{\hat{T}_i} < \infty$, for all $i = p, q$.

Explicit dilation for finite rank tuples

Theorem (–, Barik, Haria & Sarkar, 18)

If $T \in \mathcal{T}_{p,q}$ is a finite rank tuple, then T dilates to the n-tuple of commuting isometries

$$(M_{z_1}, \ldots, M_{z_{p-1}}, M_{\Phi_p}, M_{z_p}, \ldots, M_{z_{n-1}}) \text{ on } H^2_{\mathcal{D}_{\hat{T}_p}}(\mathbb{D}^{n-1}),$$

where Φ_p is an one variable inner function in $H^\infty_{\mathcal{B}(\mathcal{D}_{\hat{T}_p})}(\mathbb{D})$.

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Theorem (–, Barik, Haria & Sarkar, 18)

If $T \in \mathcal{T}_{p,q}$ is a finite rank operator, then $\exists$ an algebraic variety V in $\overline{\mathbb{D}}^n$ such that

$$\|p(T)\| \leq \sup_{\mathbf{z} \in V} |p(\mathbf{z})|, \quad (p \in \mathbb{C}[z_1, \ldots, z_n]).$$

If, in addition, T_p is a pure contraction, then $\exists$ a distinguished variety V' in $\mathbb{D}^2$ such that $V = V' \times \mathbb{D}^{n-2} \subseteq \mathbb{D}^n$.

Key ideas

Let $T \in \mathcal{T}_{p,q}$ for some $1 \leq p < q \leq n$.

- Since $\hat{T}_p$ is pure and Szegö tuple, we can dilate $\hat{T}_p$ to $(M_{z_1}, \dots, M_{z_{n-1}})$ on $H^2_{\mathcal{D}_{\hat{T}_p}}(\mathbb{D}^{n-1})$.

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- Need to find a unitary $\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{D}_{\hat{T}_p} \oplus H \to \mathcal{D}_{\hat{T}_p} \oplus H$ for some Hilbert space H such that $\Phi(z) = A + zB(I - zD)^{-1}C$ is inner and M_Φ is a dilation of T_p.

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- It turns out that the map $\{D_{\hat{T}_p} h \oplus D_{\hat{T}_q} T_q^* h : h \in \mathcal{H}\} \to \{D_{\hat{T}_p} T_p^* h \oplus D_{\hat{T}_p} h : h \in \mathcal{H}\}$ is an isometry and therefore extends to a unitary $U : \mathcal{D}_{\hat{T}_p} \oplus \mathcal{D}_{\hat{T}_q} \to \mathcal{D}_{\hat{T}_p} \oplus \mathcal{D}_{\hat{T}_q}$. This uniary do the job for us.

Dilation of general tuples

Theorem (–, Barik, Haria & Sarkar, 18)

Let $T = (T_1, \ldots, T_n) \in \mathcal{T}_{p,q}$. *Then* T *dilates to*

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on $H^2_{\mathcal{E}}(\mathbb{D}^{n-1})$ *with*

$$\Phi_p(z)\Phi_q(z) = \Phi_q(z)\Phi_p(z) = z_p I_{\mathcal{E}},$$

for some Hilbert space $\mathcal{E}$.

Let $T = (T_1, T_2, T_3) \in \mathcal{T}_{2,3}$. Then (T_1, T_3) and (T_1, T_2) satisfy Szegö positivity.

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- This can be done using a technique recently found in the following artice:
B. Krishna Das, S. Sarkar, J. Sarkar, *Factorizations of contractions*, Adv. Math. 322 (2017), 186 - 200.

References

R. E. Curto and F.-H. Vasilescu, *Standard operator models in the polydisc*, Indiana Univ. Math. J. 42 (1993), 791-810.

S. Barik, B. K. Das, K. J. Haria and J. Sarkar, *Isometric dilations and von Neumann inequality for a class of tuples in the polydisc*, Tran. Amer. Math. Soc. (to appear).

A. Grinshpan, D.S. Kaliuzhnyi-Verbovetskyi, V. Vinnikov and H.J. Woerdeman, *Classes of tuples of commuting contractions satisfying the multivariable von Neumann inequality*, J. Funct. Anal. 256 (2009), 3035-3054.

Thank You !