Linear dynamics in reproducing kernel Hilbert spaces

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Outline

Hypercyclicity and chaos

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Main motivations arise from dynamical systems (via Birkhoff's Theorem) and operator theory (via cyclic vectors)

Topological Transitivity and mixing

(Birkhoff's Transitivity Theorem 1929):

Let $g : M \to M$ be a continuous map on a separable, complete metric space with no isolated points. The g is topologically transitive if and only if it has a dense orbit.

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Clearly mixing $\Rightarrow$ transitivity, but they are different notions.

How to check if an operator is hypercyclic?

Theorem

(The Hypercyclicity Criterion)
Suppose that D is dense in X and $\{n_k\} \subseteq \mathbb{N}$ is a strictly increasing sequence. If
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then T is hypercyclic. If

$$n_k = k,$$

for all $k \in \mathbb{N}$, then T is topologically mixing.

Theorem

(The Chaoticity Criterion) T is chaotic if for each $x \in D$, there exists a sequence $\{u_k\}_{k \geq 0}$ in X with $u_0 = x$ such that

$$\sum_{n \geq 0} T^n x \qquad \text{and} \qquad \sum_{n \geq 0} u_n,$$

are unconditionally convergent, and

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Chaoticity Criterion $\Rightarrow$ Mixing Criterion, but in general chaos and topological mixing are different.

Interesting Properties

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(1) $\sigma_p(T^*)$ is always empty. (More is true)
(2) Every orbit of T^* is unbounded.
(3) If X is complex, then every connected component of $\sigma(T)$ meets S^1 in $\mathbb{C}$.
(4) T^p is hypercyclic for all $p \geq 1$ and shares the set of hypercyclic vectors
(5) so is λT for unimodular λ

- Every infinite dimensional separable Banach space supports a hypercyclic operator.

Suppose T is chaotic on a complex Banach space.
(1) The set of all perioidc vectors is the subspace generated by eigenvectors cooresponding to rational eigenvalues.
(2) $\sigma(T)$ has no isolated points

- Not every infinite dim separable Banach space supports a chaotic operator.

Examples of hypercyclic operators

Historical examples are due to Birkhoff (1929), McLane (1951) and Rolewicz (1969).

(i) The wighted backward shift

$$B_w(x_0, x_1, \ldots) = (w_1 x_1, w_2 x_2, \ldots) \text{ is hypercyclic on } \ell^p \text{ iff}$$
$$\limsup_n (w_1 w_2 \ldots w_n) = \infty.$$

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(ii) The adjoint multiplier M_φ^* on the Hardy space $H^2(\mathbb{D})$ is hypercyclic if and only if φ is non-constant and $\varphi(\mathbb{D})$ meets S^1.

(iii) Every non-trivial operator commuting the derivative operator is hypercyclic on $\mathscr{H}(\mathbb{C})$.

Examples of chaotic operators

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$B_w(x_0, x_1, \dots) = (w_1 x_1, w_2 x_2, \dots)$ is chaotic on ℓ^p iff the sequence $(w_1 w_2 \dots w_n)^{-p}$ is in ℓ^p.

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(ii) The adjoint multiplier M_φ^* on the Hardy space $H^2(\mathbb{D})$ is chaotic if and only if it is hypercyclic.
(iii) Every non-trivial operator commuting the derivative operator is chaotic on $\mathscr{H}(\mathbb{C})$.

Reproducing kernel Hilbert spaces

Scalar kernels:

Let $\mathscr{H}$ be an analytic function space over $\mathbb{D}$ and let the evaluation functonal

$$w \to f(w)$$

be bounded for each w. Then there exists $k_w \in \mathscr{H}$ such that $f(w) = \langle f, k_w \rangle$ for all f and w. The function

$$k(z, w) = \langle k_w, k_z \rangle$$

is called the (positive definite) kernel for $\mathscr{H}$. The functions k_w form a total set.

There is a one-one coresspondence between RKH spaces and positive definite kernels.

Operator valued kernels:

Let $\mathscr{E}$ be a Hilbert space. An function $K : \mathbb{D} \times \mathbb{D} \to \mathscr{B}(\mathscr{E})$ is an *analytic kernel* if K is analytic in the first variable and

$$\sum_{i,j=1}^{n} \langle K(z_i, z_j)\eta_j, \eta_i \rangle_{\mathscr{E}} \geq 0,$$

for all $\{z_i\}_{i=1}^{n} \subseteq \mathbb{D}$ and $\{\eta_i\}_{i=1}^{n} \subseteq \mathscr{E}$ and $n \in \mathbb{N}$. In this case there exists a Hilbert space $\mathscr{H}_{\mathscr{E}}(K)$ of $\mathscr{E}$-valued analytic functions on $\mathbb{D}$ such that

$$\{K(\cdot, w)\eta : w \in \mathbb{D}, \eta \in \mathscr{E}\},$$

is a total set in $\mathscr{H}_{\mathscr{E}}(K)$, where

$$(K(\cdot, w)\eta)(z) = K(z, w)\eta.$$

(i)(Reproducing property)

$$\langle f, K(\cdot, w)\eta \rangle_{\mathscr{H}_{\mathscr{E}}(K)} = \langle f(w), \eta \rangle_{\mathscr{E}}$$

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(ii)The evaluation operator $f \to f(w)$ is bounded for each w.
(iii)$K(z,w)^* = K(w,z)$
Analyticity in $\overline{w}$ and reproducing property $\Rightarrow$

$$K_n(z) = \frac{\partial^n K}{\partial \bar{w}^n}(z,0) \in \mathscr{B}(\mathscr{E})$$

and the function

$$K_{n,\eta}(z) = K_n(z)\eta$$

is in $\mathscr{H}_{\mathscr{E}}(K)$.

Derivatives: norms and connection with the adjoint operator M_z^*

$$\langle f^{(n)}(0), \eta \rangle_{\mathcal{E}} = \langle f, K_{n,\eta} \rangle_{\mathscr{H}_{\mathcal{E}}(K)},$$

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If the multiplication operator M_z is bounded on $\mathscr{H}_{\mathscr{E}}(K)$, then

$$M_z^* (\frac{1}{n!} K_{n,\eta}) = \begin{cases} \frac{1}{(n-1)!} K_{n-1,\eta} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0, \end{cases}$$

for all $\eta \in \mathscr{E}$.

Main Results

Suppose that $\mathscr{E}_0$ is dense.
(1) If

$$\liminf_n \left(\frac{1}{(n!)^2} \left\langle \left(\frac{\partial^{2n} K}{\partial z^n \partial \bar{w}^n}(0,0) \right) \eta, \eta \right\rangle_{\mathscr{E}} \right) = 0,$$

uniformly in $\eta \in \mathscr{E}_0$, then M_z^* is hypercyclic.
(2) If

$$\lim_n \left(\frac{1}{(n!)^2} \left\langle \left(\frac{\partial^{2n} K}{\partial z^n \partial \bar{w}^n}(0,0) \right) \eta, \eta \right\rangle_{\mathscr{E}} \right) = 0,$$

for all $\eta \in \mathscr{E}_0$, then M_z^* is topologically mixing.

Scalar case

Suppose M_z is bounded on $\mathscr{H}(k)$ corresponding to an analytic scalar kernel k. Then:
(1) M_z^* is hypercyclic if

$$\liminf_n \frac{1}{(n!)^2} \frac{\partial^{2n} k}{\partial z^n \partial \bar{w}^n}(0,0) = 0.$$

(2) M_z^* is topologically mixing if

$$\lim_n \frac{1}{(n!)^2} \frac{\partial^{2n} k}{\partial z^n \partial \bar{w}^n}(0,0) = 0.$$

Conditions for chaos of M_z^*

(1) Assume that M_z is bounded and $\mathscr{E}_0$ is dense in $\mathscr{E}$. Then M_z^* is chaotic on $\mathscr{H}_{\mathscr{E}}(K)$ if

$$\sum_{n,m\geq 0} \frac{1}{n!\ m!} \langle \Big(\frac{\partial^{n+m} K}{\partial z^n \partial \bar{w}^n}(0,0) \Big) \eta, \eta \rangle_{\mathscr{E}},$$

is convergent in some sense for all $\eta \in \mathscr{E}_0$.

Counter-example

Consider the diagonal kernel

$$k(z,w) = \sum_n \beta_n z^n \overline{w}^n,$$

where (β_n) is positive real sequence so that the space k has the domain the bi-disc. Let

$$K(z,w) = k(z,w) I_{\mathscr{E}}.$$

Then $\mathscr{H}_{\mathscr{E}}(K)$ is the tensor product of $\mathscr{H}(k)$ with $\mathscr{E}$. It can be shown that
M_z^* is hypercyclic iff $\inf \beta_n = 0$.
M_z^* is mixing iff $\lim_n \beta_n = 0$
It is chaotic iff $\sum_n \beta_n < \infty$.

Counter-example

Now consider a new

$$k_1(z,w) = (1-z)^{-1} k(z,w)(1-\overline{w})^{-1}.$$

It can be shown through a suitable (β_n) that M_z^* is hypercyclic on $\mathscr{H}(k_1)$, but the sufficient conditions are not necessary.

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It can be shown through a suitable (β_n) that M_z^* is hypercyclic on $\mathscr{H}(k_1)$, but the sufficient conditions are not necessary. However, if $P(z)$ is an operator valued polynomial with injective $P(0)$, then one can show that the sufficient conditions are also necessary for $\mathscr{H}_{\mathscr{E}}(K_1)$, where

$$K_1(z,w) = P(z)K(z,w)P(w)^*.$$

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Thank You