Characterization of invariant subspaces in the polydisc

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- To give a complete characterization of (joint) invariant subspaces for $(M_{z_1}, \dots, M_{z_n})$ on the Hardy space $H^2(\mathbb{D}^n)$ over the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$, $n > 1$.

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- To discuss about a complete set of unitary invariants for invariant subspaces as well as unitarily equivalent invariant subspaces.
- To classify a large class of n-tuples of commuting isometries.

Characterization of invariant subspaces . . .

Notation

- Unit polydisc $\mathbb{D}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| < 1, i = 1, \ldots, n\}$.
- Hardy space $H^2(\mathbb{D}) = \{f = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$.
- Vector-valued Hardy space

$$H^2_{\mathcal{E}}(\mathbb{D}) = \{f = \sum_{n=0}^{\infty} a_n z^n : a_n \in \mathcal{E} \text{ and } \sum_{n=0}^{\infty} \|a_n\|^2_{\mathcal{E}} < \infty\},$$

  where $\mathcal{E}$ is some Hilbert space.
- M_z denote the multiplication operator on $H^2_{\mathcal{E}}(\mathbb{D})$ defined by

$$(M_z f)(w) = wf(w) \quad (f \in H^2_{\mathcal{E}}(\mathbb{D}), w \in \mathbb{D}).$$

- $H^{\infty}_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ denote the space of bounded $\mathcal{B}(\mathcal{E})$-valued holomorphic functions on $\mathbb{D}$.

Invariant subspaces: Motivation

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- Given an invariant subspace $\mathcal{M}$ of $T \in \mathcal{B}(\mathcal{H})$, one can view T as an operator matrix with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$

$$\begin{bmatrix} T|_{\mathcal{M}} & * \\ 0 & * \end{bmatrix}.$$

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- One of the most famous open problems in operator theory and function theory is the so-called invariant subspace problem: Does every bounded linear operator have a non-trivial closed invariant subspace?

Invariant subspaces: Motivation

- The celebrated Beurling theorem (1949) says that a non-zero closed subspace $\mathcal{S}$ of $H^2(\mathbb{D})$ is invariant for M_z if and only if there exists an inner function $\theta \in H^\infty(\mathbb{D})$ such that
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- Rudin's pathological examples (Rudin (1969)): There exist invariant subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ for (M_{z_1}, M_{z_2}) on $H^2(\mathbb{D}^2)$ such that
(1) $\mathcal{S}_1$ is not finitely generated, and
(2) $\mathcal{S}_2 \cap H^\infty(\mathbb{D}^2) = \{0\}$.

Invariant subspaces: Motivation

- To understand the structure of a large class of operators we need to understand the structure of shift invariant subspaces for the vector-valued Hardy space over the unit disc.

Theorem (Beurling-Lax-Halmos)

A non-zero closed subspace $\mathcal{S}$ of $H^2_{\mathcal{E}}(\mathbb{D})$ is invariant for M_z if and only if there exists a closed subspace $\mathcal{F} \subseteq \mathcal{E}$ and an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{F},\mathcal{E})}(\mathbb{D})$ such that

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Idea

- Identify Hardy space over polydisc $H^2(\mathbb{D}^{n+1})$ to the $H^2(\mathbb{D}^n)$-valued Hardy space over disc $H^2_{H^2(\mathbb{D}^n)}(\mathbb{D})$.

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Idea

- Identify Hardy space over polydisc $H^2(\mathbb{D}^{n+1})$ to the $H^2(\mathbb{D}^n)$-valued Hardy space over disc $H^2_{H^2(\mathbb{D}^n)}(\mathbb{D})$.
- Represent $(M_{z_1}, M_{z_2}, \ldots, M_{z_{n+1}})$ on $H^2(\mathbb{D}^{n+1})$ to $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{H^2(\mathbb{D}^n)}(\mathbb{D})$, where $\kappa_i \in H^\infty_{\mathcal{B}(H^2(\mathbb{D}^n))}(\mathbb{D})$, $i = 1, \ldots, n$, is a constant as well as simple and explicit $\mathcal{B}(H^2(\mathbb{D}^n))$-valued analytic function.

Basic definitions

- A closed subspace $\mathcal{S} \subseteq H^2_{\mathcal{E}}(\mathbb{D}^n)$ is called a *(joint) invariant subspace* for $(M_{z_1}, \ldots, M_{z_n})$ on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ if
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- Let V be a pure isometry on $\mathcal{H}$. Then
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where $\mathcal{W} = \ker V^* = \mathcal{H} \ominus V\mathcal{H}$.
- The natural map $\Pi_V : \mathcal{H} \to H^2_{\mathcal{W}}(\mathbb{D})$ defined by
$$\Pi_V(V^m \eta) = z^m \eta,$$
for all $m \geq 0$ and $\eta \in \mathcal{W}$, is a unitary operator and
$$\Pi_V V = M_z \Pi_V.$$
We call Π_V the *Wold-von Neumann decomposition* of the shift V.

Commutator of shift

Theorem 1 (Maji, Sarkar, & Sankar' 18)

Let V be a pure isometry on $\mathcal{H}$, and let C be a bounded operator on $\mathcal{H}$. Let Pi_V be the Wold-von Neumann decomposition of V. Let $\mathcal{W} = Ker(V^*)$ and assume that $M = \Pi_V C \Pi_V^*$. Then

$$CV = VC,$$

if and only if

$$M = M_\Theta,$$

where

$$\Theta(z) = P_{\mathcal{W}}(I_{\mathcal{H}} - zV^*)^{-1} C \mid_{\mathcal{W}} \qquad (z \in \mathbb{D}).$$

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Remark

In addition if $CV^* = V^*C$, then

$$\Theta(z) = C|_{\mathcal{W}} = \Theta(0) \qquad (z \in \mathbb{D}),$$

as $C(I - VV^*) = (I - VV^*)C$ and $V^{*m}|_{\mathcal{W}} = 0$ for all $m \geq 1$.

Preparation for main result

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- We now identify $H^2_{\mathcal{E}}(\mathbb{D}^2)$ with $H^2_{H^2_{\mathcal{E}}(\mathbb{D})}(\mathbb{D})$ by the canonical unitaries

$$H^2_{\mathcal{E}}(\mathbb{D}^2) \xrightarrow{\hat{U}} H^2(\mathbb{D}) \otimes H^2_{\mathcal{E}}(\mathbb{D}) \xrightarrow{\tilde{U}} H^2_{H^2_{\mathcal{E}}(\mathbb{D})}(\mathbb{D})$$

where

$$\hat{U}(z_1^{k_1} z_2^{k_2} \eta) = z^{k_1} \otimes (z_1^{k_2} \eta), \qquad (\ k_1, k_2 \geq 0,\ \eta \in \mathcal{E}\ )$$

and

$$\tilde{U}(z^k \otimes \zeta) = z^k \zeta, \qquad (\ k \geq 0,\ \zeta \in H^2_{\mathcal{E}}(\mathbb{D})\ ).$$

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- Set $U = \tilde{U}\hat{U}$. Then it follows that $U : H^2_{\mathcal{E}}(\mathbb{D}^2) \to H^2_{H^2_{\mathcal{E}}(\mathbb{D})}(\mathbb{D})$ is a unitary operator. Since

$$\hat{U} M_{z_1} = (M_z \otimes I_{H^2_{\mathcal{E}}(\mathbb{D})})\hat{U} \quad \text{and} \quad \hat{U} M_{z_2} = (I_{H^2(\mathbb{D})} \otimes M_{z_1})\hat{U},$$

we have $UM_{z_1} = M_z U$, and $UM_{z_2} = M_{\kappa_1} U$, where $\kappa_1(w) = M_{z_1}$ for $w \in \mathbb{D}$.

Preparation for main result

Let $\mathcal{E}$ be a Hilbert space and let $\mathcal{E}_n = H^2(\mathbb{D}^n) \otimes \mathcal{E}$ for $n \geq 1$. Let $\kappa_i \in H^\infty_{\mathcal{B}(\mathcal{E}_n)}(\mathbb{D})$ denote the $\mathcal{B}(\mathcal{E}_n)$-valued constant function on $\mathbb{D}$ defined by

$$\kappa_i(w) = M_{z_i} \in \mathcal{B}(\mathcal{E}_n),$$

for all $w \in \mathbb{D}$, and let M_{κ_i} denote the multiplication operator on $H^2_{\mathcal{E}_n}(\mathbb{D})$ defined by

$$M_{\kappa_i} f = \kappa_i f,$$

for all $f \in H^2_{\mathcal{E}_n}(\mathbb{D})$ and $i = 1, \ldots, n$. Then

Theorem 2 (Maji, Aneesh, Sarkar, & Sankar' 18)

(i) $(M_{z_1}, M_{z_2} \ldots, M_{z_{n+1}})$ on $H^2_{\mathcal{E}}(\mathbb{D}^{n+1})$ and $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent.

Preparation for main result

- Let $\mathcal{S} \subseteq H^2_{H^2_{\mathcal{E}}(\mathbb{D})}(\mathbb{D})$ be a closed invariant subspace for (M_z, M_{κ_1}) on $H^2_{H^2_{\mathcal{E}}(\mathbb{D})}(\mathbb{D})$. Set
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$$V = M_z|_{\mathcal{S}} \quad \text{and} \quad V_1 = M_{\kappa_1}|_{\mathcal{S}}.$$
- Let $\Pi_V : \mathcal{S} \to H^2_{\mathcal{W}}(\mathbb{D})$ be the Wold-von Neumann decomposition of V on $\mathcal{S}$. Then
$$\Pi_V V \Pi_V^* = M_z \quad \text{and} \quad \Pi_V V_1 \Pi_V^* = M_{\Phi_1},$$
where
$$\Phi_1(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - wV^*)^{-1} V_1|_{\mathcal{W}},$$
for all $w \in \mathbb{D}$, $\Phi_1 \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$.

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where
$$\Phi_1(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - wV^*)^{-1} V_1|_{\mathcal{W}},$$
for all $w \in \mathbb{D}$, $\Phi_1 \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$.
- Let $i_{\mathcal{S}}$ denote the inclusion map $i_{\mathcal{S}} : \mathcal{S} \hookrightarrow H^2_{\mathcal{E}_n}(\mathbb{D})$. Then
$$H^2_{\mathcal{W}}(\mathbb{D}) \xrightarrow{\Pi_V^*} \mathcal{S} \xrightarrow{i_{\mathcal{S}}} H^2_{H^2_{\mathcal{E}}(\mathbb{D})}(\mathbb{D})$$
i.e., $\Pi_{\mathcal{S}} = i_{\mathcal{S}} \circ \Pi_V^* : H^2_{\mathcal{W}}(\mathbb{D}) \to H^2_{H^2_{\mathcal{E}}(\mathbb{D})}(\mathbb{D})$ is an isometry and
$$\operatorname{ran} \Pi_{\mathcal{S}} = \mathcal{S}.$$

Main result

We have invariant subspace result on vector-valued Hardy space over polydisc setting:

Theorem 2 (Maji, Aneesh, Sarkar, & Sankar' 18)

(ii) Let $\mathcal{E}$ be a Hilbert space, $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ be a closed subspace, and let $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$. Then $\mathcal{S}$ is invariant for $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ if and only if $(M_{\Phi_1}, \dots, M_{\Phi_n})$ is an *n-tuple of commuting shifts* on $H^2_{\mathcal{W}}(\mathbb{D})$ and there exists an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}, \mathcal{E}_n)}(\mathbb{D})$ such that

$$\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D}),$$

and

$$\kappa_i \Theta = \Theta \Phi_i,$$

where

$$\Phi_i(w) = P_{\mathcal{W}}(I_{\mathcal{S}} - wP_{\mathcal{S}}M_z^*)^{-1} M_{\kappa_i}|_{\mathcal{W}},$$

for all $w \in \mathbb{D}$ and $i = 1, \dots, n$.

Remarks

- One obvious necessary condition for a closed subspace $\mathcal{S}$ of $H^2_{\mathcal{E}_n}(\mathbb{D})$ to be (joint) invariant for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ is that $\mathcal{S}$ is invariant for M_z, and, consequently
$$\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D}),$$
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- Again $\kappa_i \mathcal{S} \subseteq \mathcal{S}, \implies$
$$\kappa_i \Theta = \Theta \Gamma_i,$$
for some $\Gamma_i \in \mathcal{B}(H^2_{\mathcal{W}}(\mathbb{D}))$, $i = 1, \ldots, n$ (by Douglas's range inclusion theorem).

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- Again $\kappa_i \mathcal{S} \subseteq \mathcal{S}, \implies$
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for some $\Gamma_i \in \mathcal{B}(H^2_{\mathcal{W}}(\mathbb{D}))$, $i = 1, \ldots, n$ (by Douglas's range inclusion theorem).
- In the above theorem, we prove that Γ_i is explicit, that is
$$\Gamma_i = \Phi_i \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D}),$$
for all $i = 1, \ldots, n$, and $(\Gamma_1, \ldots, \Gamma_n)$ is an n-tuple of commuting shifts on $H^2_{\mathcal{W}}(\mathbb{D})$.

Remarks

Uniqueness

Let $\mathcal{S}$ be an invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$. Then $\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D})$ and

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from the above Theorem.

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$$\kappa_i \Theta = \Theta \Phi_i \qquad (i = 1, \ldots, n),$$

from the above Theorem.

Now suppose $\mathcal{S} = \tilde{\Theta} H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$ and $\kappa_i \tilde{\Theta} = \tilde{\Theta}\tilde{\Phi}_i$ for some Hilbert space $\tilde{\mathcal{W}}$, inner function $\tilde{\Theta} \in H^\infty_{\mathcal{B}(\tilde{\mathcal{W}})}(\mathbb{D})$ and shift $M_{\tilde{\Phi}_i}$ on $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$, $i = 1, \ldots, n$.

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Then there exists a *unitary operator* $\tau : \mathcal{W} \to \tilde{\mathcal{W}}$ such that

$$\Theta = \tilde{\Theta}\tau,$$

and

$$\tau \Phi_i = \tilde{\Phi}_i \tau,$$

for all $i = 1, \ldots, n$.

Nested invariant subspaces

Theorem (Maji, Aneesh, Sarkar, & Sankar' 18)

Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{S}_1 = \Theta_1 H^2_{\mathcal{W}_1}(\mathbb{D})$ and $\mathcal{S}_2 = \Theta_2 H^2_{\mathcal{W}_2}(\mathbb{D})$ be two invariant subspaces for $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $\mathcal{W}_j = \mathcal{S}_j \ominus z\mathcal{S}_j$ for $j = 1, 2$. Let

$$\Phi_{j,i}(w) = P_{\mathcal{W}_j}(I_{\mathcal{S}_j} - wP_{\mathcal{S}_j} M_z^*)^{-1} M_{\kappa_i}|_{\mathcal{W}_j},$$

for all $w \in \mathbb{D}$, $j = 1, 2$, and $i = 1, \dots, n$.

Nested invariant subspaces

Theorem (Maji, Aneesh, Sarkar, & Sankar' 18)

Let $\mathcal{E}$ be a Hilbert space, and let $\mathcal{S}_1 = \Theta_1 H^2_{\mathcal{W}_1}(\mathbb{D})$ and $\mathcal{S}_2 = \Theta_2 H^2_{\mathcal{W}_2}(\mathbb{D})$ be two invariant subspaces for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $\mathcal{W}_j = \mathcal{S}_j \ominus z\mathcal{S}_j$ for $j = 1, 2$. Let

$$\Phi_{j,i}(w) = P_{\mathcal{W}_j}(I_{\mathcal{S}_j} - wP_{\mathcal{S}_j} M_z^*)^{-1} M_{\kappa_i}|_{\mathcal{W}_j},$$

for all $w \in \mathbb{D}$, $j = 1, 2$, and $i = 1, \ldots, n$.
Then $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if there exists an inner multiplier $\Psi \in H^\infty_{\mathcal{B}(\mathcal{W}_1, \mathcal{W}_2)}(\mathbb{D})$ such that $\Theta_1 = \Theta_2 \Psi$ and $\Psi\Phi_{1,i} = \Phi_{2,i}\Psi$ for all $i = 1, \ldots, n$.

Unitarily equivalent invariant subspaces

Definition

Let $\mathcal{S}$ and $\tilde{\mathcal{S}}$ be invariant subspaces for the $(n+1)$-tuples of multiplication operators $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $H^2_{\tilde{\mathcal{E}}_n}(\mathbb{D})$, respectively. We say that $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are *unitarily equivalent*, and write $\mathcal{S} \cong \tilde{\mathcal{S}}$, if there is a unitary map $U : \mathcal{S} \to \tilde{\mathcal{S}}$ such that

$$UM_z|_{\mathcal{S}} = M_z|_{\tilde{\mathcal{S}}}U \quad \text{and} \quad UM_{\kappa_i}|_{\mathcal{S}} = M_{\kappa_i}|_{\tilde{\mathcal{S}}}U,$$

for all $i = 1, \ldots, n$.

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for all $i = 1, \ldots, n$.

Identification

There exists a unitary operator $U_{\mathcal{E}} : H^2_{\mathcal{E}}(\mathbb{D}^{n+1}) \to H^2_{\mathcal{E}_n}(\mathbb{D})$ such that

$$U_{\mathcal{E}} M_{z_1} = M_z U_{\mathcal{E}},$$

and

$$U_{\mathcal{E}} M_{z_{i+1}} = M_{\kappa_i} U_{\mathcal{E}},$$

for all $i = 1, \ldots, n$.

Intertwining maps

Let $\mathcal{F}$ be another Hilbert space, and let $X : H^2_{\mathcal{E}}(\mathbb{D}^{n+1}) \to H^2_{\mathcal{F}}(\mathbb{D}^{n+1})$ be a bounded linear operator such that

$$XM_{z_i} = M_{z_i}X, \tag{0.1}$$

for all $i = 1, \ldots, n+1$. Set

$$X_n = U_{\mathcal{F}} X U_{\mathcal{E}}^*.$$

Then $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ is bounded and

$$X_n M_z = M_z X_n \quad \text{and} \quad X_n M_{\kappa_i} = M_{\kappa_i} X_n, \tag{0.2}$$

for all $i = 1, \ldots, n$.

Definition

Any map satisfying (0.2) will be referred to *module maps*.

Unitarily equivalent invariant subspaces

Theorem (Maji, Aneesh, Sarkar, & Sankar' 18)

Let $\mathcal{S} \subseteq H^2_{\mathcal{F}_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{F}_n}(\mathbb{D})$. Then $\mathcal{S} \cong H^2_{\mathcal{E}_n}(\mathbb{D})$ if and only if there exists an isometric module map $X_n : H^2_{\mathcal{E}_n}(\mathbb{D}) \to H^2_{\mathcal{F}_n}(\mathbb{D})$ such that

$$\mathcal{S} = X_n H^2_{\mathcal{E}_n}(\mathbb{D}).$$

Moreover, in this case

$$\dim \mathcal{E} \leq \dim \mathcal{F}.$$

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Moreover, in this case

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Corollary

Let $\mathcal{S} \subseteq H^2_{H_n}(\mathbb{D})$ be a closed invariant subspace for $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{H_n}(\mathbb{D})$. Then $\mathcal{S} \cong H^2_{H_n}(\mathbb{D})$ if and only if there exists an isometric module map $X_n : H^2_{H_n}(\mathbb{D}) \to H^2_{H_n}(\mathbb{D})$ such that

$$\mathcal{S} = X_n(H^2_{H_n}(\mathbb{D})).$$

The above corollary was first observed by Agrawal, Clark and Douglas (1986).

A complete set of unitary invariants

Definition

Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be Hilbert spaces, and let $\{\Psi_1, \dots, \Psi_n\} \subseteq H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ and $\{\tilde{\Psi}_1, \dots, \tilde{\Psi}_n\} \subseteq H^\infty_{\mathcal{B}(\tilde{\mathcal{E}})}(\mathbb{D})$. We say that $\{\Psi_1, \dots, \Psi_n\}$ and $\{\tilde{\Psi}_1, \dots, \tilde{\Psi}_n\}$ coincide if there exists a unitary operator $\tau : \mathcal{E} \to \tilde{\mathcal{E}}$ such that

$$\tau \Psi_i(z) = \tilde{\Psi}_i(z) \tau,$$

for all $z \in \mathbb{D}$ and $i = 1, \dots, n$.

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Theorem (Maji, Aneesh, Sarkar, & Sankar' 18)

Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be Hilbert spaces. Let $\mathcal{S} \subseteq H^2_{\mathcal{E}_n}(\mathbb{D})$ and $\tilde{\mathcal{S}} \subseteq H^2_{\tilde{\mathcal{E}}_n}(\mathbb{D})$ be invariant subspaces for $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ and $H^2_{\tilde{\mathcal{E}}_n}(\mathbb{D})$, respectively. Then $\mathcal{S} \cong \tilde{\mathcal{S}}$ if and only if $\{\Phi_1, \dots, \Phi_n\}$ and $\{\tilde{\Phi}_1, \dots, \tilde{\Phi}_n\}$ coincide.

Representations of model isometries

Question

Given a Hilbert space $\mathcal{E}$, characterize $(n+1)$-tuples of commuting shifts on Hilbert spaces that are unitarily equivalent to $(M_z, M_{\kappa_1}, \ldots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$.

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Answer

Answer to this question is related to (numerical invariant) the rank of an operator associated with the Szegö kernel on $\mathbb{D}^{n+1}$.

Representations of model isometries

Definition

The *defect operator* corresponding to an m-tuple of commuting contractions $(T_1, \ldots, T_m)$ on a Hilbert space $\mathcal{H}$ is defined (see, Guo & Yang (2004)) as

$$\mathbb{S}_m^{-1}(T_1, \ldots, T_m) = \sum_{0 \leq |\mathbf{k}| \leq m} (-1)^{|\mathbf{k}|} T_1^{k_1} \cdots T_m^{k_m} T_1^{*k_1} \cdots T_m^{*k_m},$$

where $|k| = k_1 + k_2 + \ldots + k_m$, $0 \leq k_i \leq 1$, $i = 1, \ldots, m$.

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where $|k| = k_1 + k_2 + \dots + k_m$, $0 \le k_i \le 1$, $i = 1, \dots, m$.

Definition

We say that $(T_1, \dots, T_m)$ is of *rank* p ($p \in \mathbb{N} \cup \{\infty\}$) if

$$\text{rank } [\mathbb{S}_m^{-1}(T_1, \dots, T_m)] = p,$$

and we write

$$\text{rank } (T_1, \dots, T_m) = p.$$

Representations of model isometries

- Let $(V, V_1 \ldots, V_n)$ be an $(n+1)$-tuple of doubly commuting shifts on $\mathcal{H}$. Then Sarkar (2014) proved that $(V, V_1, \ldots, V_n)$ on $\mathcal{H}$ and $(M_{z_1}, \ldots, M_{z_{n+1}})$ on $H^2_{\mathcal{D}}(\mathbb{D}^{n+1})$ are unitarily equivalent, where

$$\mathcal{D} = \text{ran } \mathbb{S}_{n+1}^{-1}(V, V_1, \ldots, V_n) = \left(\bigcap_{i=1}^{n} \ker V_i^*\right) \cap \ker V^*.$$

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Theorem (Maji, Aneesh, Sarkar, & Sankar' 18)

Let $(V, V_1, \dots, V_n)$ be an $(n+1)$-tuple of doubly commuting shifts on some Hilbert space $\mathcal{H}$. Let $\mathcal{W} = \mathcal{H} \ominus V\mathcal{H}$, and let

$$\Psi_i(z) = V_i|_{\mathcal{W}} \qquad (i = 1, \dots, n),$$

for all $z \in \mathbb{D}$. Then $(V, V_1, \dots, V_n)$ on $\mathcal{H}$, $(M_z, M_{\Psi_1}, \dots, M_{\Psi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$, and $(M_z, M_{\kappa_1}, \dots, M_{\kappa_n})$ on $H^2_{\mathcal{E}_n}(\mathbb{D})$ are unitarily equivalent, where $\mathcal{E}$ is a Hilbert space and $\dim \mathcal{E} = \text{rank } (V, V_1, \dots, V_n)$.

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Thank You!