Entropy for general quantum systems

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Outline

1. Classical roots
2. Algebraic preliminaries
3. Relative entropy for general systems
4. Continuous entropy for general systems

States with entropy

What exactly is entropy?

In the modern microscopic interpretation of entropy in statistical mechanics, entropy is the amount of additional information needed to specify the exact physical state of a system, given its thermodynamic specification.

It is often said that entropy is an expression of the disorder, or randomness of a system, or of our lack of information about it.

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Boltzmann

Spatially homogeneous Boltzmann equation (for the dynamics of rarefied gases) [1872]:

$$\frac{\partial f_1}{\partial t} = \int d\Omega \int d^3 v_2 I(g,\theta)|\mathbf{v}_2 - \mathbf{v}_1|(f_1' f_2' - f_1 f_2) \qquad (1)$$

where $f_1 \equiv f(\mathbf{v}_1, t)$, $f_2' \equiv f(\mathbf{v}_2', t)$, ..., are velocity distribution functions, $I(g,\theta)$ denotes the differential scattering cross section, $d\Omega$ the solid angle element, and $g = |\mathbf{v}|$.

The natural Lyapunov functional for this equation is

$$H_+(f) = \int f(x) \log f(x) dx.$$

The (classical) continuous entropy S differs from the functional H only by sign. Hence Boltzmann's H-functional may be viewed as the first formalisation of the concept of entropy.

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Relative entropy

Let μ and ν be probability measures over a set X, and assume that $\mu \ll \nu$. The relative entropy of the states represented by μ and ν, is defined as

$$S(\mu|\nu) = \int_X \log\left(\frac{d\mu}{d\nu}\right) d\mu = \int_X \frac{d\mu}{d\nu} \log\left(\frac{d\mu}{d\nu}\right) d\nu, \tag{2}$$

provided that the integrals in the above formulae exist.

If both ν and μ are absolutely continuous with respect to the reference measure λ, then under some mild additional assumptions, this can be rewritten as

$$S(\mu|\nu) = \int_X p \log \frac{p}{q} d\lambda = \int_X p \log p - p \log q \, d\lambda, \tag{3}$$

where $p = \frac{d\mu}{d\lambda}$ and $q = \frac{d\nu}{d\lambda}$.

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Orlicz functions and spaces

Orlicz function: A convex function $\Psi : [0, \infty) \to [0, \infty]$ satisfying

- $\Psi(0) = 0$ and $\lim_{u \to \infty} \Psi(u) = \infty$,
- neither identically zero nor infinite valued on all of $(0, \infty)$,
- left continuous at $b_\Psi = \sup\{u > 0 : \Psi(u) < \infty\}$.

Definition

$f \in L^0$ belongs to $L^\Psi \Leftrightarrow \psi(\lambda|f|)$ is integrable for some $\lambda = \lambda(f) > 0$.

Luxemburg-Nakano norm: $\|f\|_\psi = \inf\{\lambda > 0 : \|\psi(|f|/\lambda)\|_1 \leq 1\}$.

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Weights and modular automorphisms

Every von Neumann algebra M admits a faithful normal semifinite weight ν.

Any fns weight ν, is a bit like an energy potential in that it induces a canonical one-parameter group of $*$-automorphisms $\sigma_t^\nu : M \to M$ $(t \in \mathbb{R})$ on M.

If an fns weight ν satisfies $\nu(a^*a) = \nu(aa^*)$ for all $a \in M$, we call ν a trace. In this case the automorphism group σ_t^ν $(t \in \mathbb{R})$ is trivial.

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Quantum paradigm: tracial case

Strategy: Let M be a von Neumann algebra equipped with a faithful normal semifinite trace τ. We then simply replace L^∞ by M, and $\int \cdot d\nu$ by τ, and see what happens if we try to copy the classical theory.

Example: Consider the case where $M = M_n(\mathbb{C})$ and $\tau = \mathrm{Tr}$:

- $L^p(M_n(\mathbb{C}), \mathrm{Tr})$ is just $M_n(\mathbb{C})$ equipped with the norm $\mathrm{Tr}(|a|^p)^{1/p}$.
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The strange ways of type III L^p spaces: 1

The challenge: Some von Neuman algebras have no fns trace, but to be true to our mission, we need to show that the earlier structures hold for these algebras as well.

An indirect construction: Let $M = L^\infty(X, \Sigma, \nu)$. Now pass to

$$A = L^\infty(X, \Sigma, \nu) \otimes L^\infty(\mathbb{R})$$

Equip A with the "trace" $\tau_A = \int \cdot \, d\nu \otimes \int_{\mathbb{R}} \cdot e^{-t} dt$ and pass to $\widetilde{A}$ (the completion of A in the topology of convergence in measure determined by τ).

(Haagerup, 1979) For any measurable function f on X (finite ν-almost everywhere) we have that

$$f \in L^p(X, \Sigma, \nu) \quad \Leftrightarrow \quad f \otimes e^{(\cdot)/p} \in \widetilde{A}.$$

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$$f \in L^p(X, \Sigma, \nu) \quad \Leftrightarrow \quad f \otimes e^{(\cdot)/p} \in \widetilde{A}.$$

The strange ways of type III L^p spaces: 2

Commutative	Quantum
Embed $L^\infty(X, \Sigma, \nu)$ into $L^\infty(X, \Sigma, \nu) \otimes L^\infty(\mathbb{R})$	"Enlarge" M equipped with an fns weight ν, by passing to $A = M \rtimes_\nu \mathbb{R}$
$\theta_s(f \otimes g)(x, t) = f(x)g(t - s)$	There exists a dual action of $\mathbb{R}$ on A in the form of a group of *-auto-morphisms $\{\theta_s\}$ $(s \in \mathbb{R})$
$\int \cdot \, d\nu \otimes \int_{\mathbb{R}} \cdot e^{-t} dt$	A admits a canonical trace τ_A characterised by $\tau_A \circ \theta_s = e^{-s}\tau_A$.

By analogy with the classical setting, we may define

$$L^p(M) = \{a \in \widetilde{A} : \theta_s(a) = e^{-s/p} a \text{ for all } s \in \mathbb{R}\} \quad \text{(Haagerup 1979)}.$$

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Type III Orlicz spaces

Haagerup's construction of L^p-spaces for type III von Neumann algebras can be extended to also allow for the construction of Orlicz spaces. (L, 2014)

The classical roots of the construction: Let $M = L^\infty(X, \Sigma, \nu)$, and let $A = L^\infty(X, \Sigma, \nu) \otimes L^\infty(\mathbb{R})$ be as before.

Given an Orlicz function Ψ, define $\varphi_\Psi : [0, \infty) \to [0, \infty)$ by

$$\varphi_\Psi(t) = \frac{1}{\Psi^{-1}(1/t)}.$$

For any measurable function f on X, we then have that

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A few nuts and bolts

The noncommutative space $L^\Psi(M)$ consists of all τ-measurable operators f affiliated to A for which we have that $\theta_s(a) = g_s^{1/2} a g_s^{1/2}$ for all $s \in \mathbb{R}$, where $g_s = \varphi_\Psi(e^{-s}h)\varphi_\Psi(h)^{-1}$.

Any state $\omega \in M_*$ induces a "dual" state $\widetilde{\omega}$ on $A = M \rtimes_\nu \mathbb{R}$.

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Cocycle derivatives 1

Theorem (Comparing weights)

Let M be a von Neumann algebra and ϕ, ψ faithful semifinite normal weights on M. Then there exists a σ-strongly continuous one parameter family $\{u_t\}$ of unitaries in M with the following properties:

- $u_{t+t'} = u_t\sigma_t^{\phi}(u_{t'})$, *for all* $t, t' \in \mathbb{R}$,
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Cocycle derivatives 2

Definition

The family of unitaries described by the above theorem is called the cocycle derivative of φ with respect to ψ and is denoted by

$$(D\varphi : D\psi)_t = u_t. \tag{4}$$

We propose:

Definition

Let ϑ, ψ be faithful normal states on M. We define the relative entropy $S(\vartheta|\psi)$ to be $S(\vartheta|\psi) = \lim_{t\to 0} \frac{-i}{t}\vartheta[(D\vartheta : D\psi)_t - \mathbb{1}]$ if the limit exists, and assign a value of ∞ to $S(\vartheta|\psi)$ otherwise.

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Araki style relative entropy

Let M be a σ-finite von Neumann algebra in standard form, and let ψ and ϕ be two faithful normal states with unit vector representatives $\Psi, \Phi \in \mathcal{H}$. (So $\psi(a) = \langle a\Psi, \Psi\rangle$ for all $a \in M$, etc.)

Tomita-Takesaki theory easily extends to show that $S_{\phi,\psi} : a\Psi \to a^*\Phi$ is a closable densely-defined anti-linear operator. The operator $\Delta_{\phi,\psi}$ is then defined to be the modulus of the closure of $S_{\phi,\psi}$.

The "standard" modular operator is used to generate the modular automorphism group of a given state. Similarly this operator then encodes the manner in which the dynamics determined by the modular automorphism group of one state, differs from the other. In view of this, Araki defined the relative entropy of ψ and ϕ to be $-\langle\Psi, \log(\Delta_{\phi,\psi})\Psi\rangle$.

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Shades of the classical formula

Definition (Kosaki)

For normal weights ϑ and ϕ on M, and some positive δ, we say that $\vartheta \leq \phi(\delta)$ if $t \to (D\vartheta : D\phi)_t = u_t$ extends to a bounded M-valued point to weak*-continuous map $z \to u_z$ on the closed strip $-\delta \leq \Im(z) \leq 0$, which is analytic on $-\delta < \Im(z) < 0$.

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Theorem

Let M be a σ-finite von Neumann algebra in standard form, and let ϑ and ϕ be two faithful normal states corresponding to the elements h_ϑ, h_ϕ of $L^1(M)$. If $\phi \leq \vartheta(\delta)$, then

$$\lim_{s \nearrow 1} tr(h_\vartheta^s \cdot \log h_\vartheta \cdot h_\phi^{1-s} - h_\vartheta^s \cdot \log h_\phi \cdot h_\phi^{1-s})$$

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Proposition (L, Majewski; 2014)

Let M be a semifinite algebra and $f \in L^1 \cap L\log(L+1)(M,\tau)$ with $f \geq 0$. Then $\tau(f\log(f+\epsilon))$ is well defined for any $\epsilon > 0$. Moreover

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is bounded above, and if in addition $f \in L^{1/2}$, it is also bounded from below.

Here $L\log(L+1)(M,\tau)$ is the Orlicz space corresponding to the function $\Psi(t) = t\log(t+1)$.

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For any $f \in [L\log(L+1) \cap L^1]$ with $f \geq 0$,

$$\int f \log(f+\epsilon)\, d\nu = \epsilon \left[\int (f/\epsilon) \log((f/\epsilon)+1)\, d\nu \right] + \log(\epsilon) \int f\, d\nu.$$

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The classical space $[L\log(L+1) \cap L^1](X, \Sigma, \nu)$ is an Orlicz space which we denote by L^{ent}.

The noncommutative space $L^{ent}(M)$ canonically embeds into both $L^1(M)$ and $L\log(L+1)(M)$ by means of the prescriptions

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Continuous entropy for general systems 1

Definition

A state ϑ on the von Neumann algebra M is called *regular* if for some element g of $L^{ent}(M)^+$, $\frac{d\widetilde{\vartheta}}{d\tau}$ is of the form $\iota_1(g)$. For such a regular state we then define the continuous entropy $\tilde{S}(\vartheta)$ to be

$$\inf_{\epsilon>0}[\epsilon\tau(\chi_{(\epsilon,\infty)}(\iota_{\log}(g))) + \log(\epsilon)\|\iota_1(g)\|_1].$$

(Here h is the density $\frac{d\widetilde{\nu}}{d\tau}$ of the dual weight $\widetilde{\nu}$.)

Continuous entropy for general systems 2

Theorem

If ϑ is a regular state, then $\tilde{S}(\vartheta)$ is well defined (although possibly infinite valued).

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Let ϑ be a regular state with $\frac{d\tilde{\vartheta}}{d\tau}$ of the form $\iota_1(g)$, where $g \in L^{ent}(M)^+$ commutes with $h = \frac{d\tilde{\nu}}{d\tau}$. (Here ν is the "reference state".) Then $\tilde{S}(\vartheta) = S(\vartheta|\nu)$.

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Selected References

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