KMS states and groupoid C*-algebras

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Plan

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Definition of KMS states

According to Gibbs and Boltzman, Gibbs states are the equilibrium states in statistical mechanics. Kubo, Martin and Schwinger have discovered a C*-algebraic formulation of Gibbs states.

Definition

Let A be a C*-algebra, σ_t a strongly continuous one-parameter group of automorphisms of A and $\beta \in \mathbb{R}$. One says that a state φ of A is (σ, β)-KMS if

1. it is invariant under σ_t for all $t \in \mathbb{R}$ and
2. for all $a, b \in A$, there exists a function F bounded and continuous on the strip $0 \leq Imz \leq \beta$ and analytic on $0 < Imz < \beta$ such that:
   - $F(t) = \varphi(a\sigma_t(b))$ for all $t \in \mathbb{R}$;
   - $F(t + i\beta) = \varphi(\sigma_t(b)a)$ for all $t \in \mathbb{R}$.

Properties of KMS states

Here A is a separable C*-algebra and $\sigma = (\sigma_t)$ is a strongly continuous one-parameter group of automorphisms of A.

- The σ_t invariance is implied by KMS$_\beta$ for $\beta \neq 0$.
- For a given β, the set Σ_β of KMS$_\beta$-states is a Choquet simplex of A^*: i.e. it is a $*$-weakly closed convex subset of A^* and every KMS$_\beta$-state is the barycenter of a unique probability measure supported on the extremal KMS$_\beta$-states.
- The extremal KMS$_\beta$-states are factorial.

KMS Problem. Determine all the KMS-states of a given dynamical system (A, σ). The discontinuities of the map $\beta \mapsto \Sigma_\beta$ are interpreted as phase transitions.

Groupoid C*-algebras

All our results about the KMS problem are restricted to diagonal automorphism groups. Let me explain what it means.

I first recall that, given a locally compact groupoid G with Haar system, we can construct a C*-algebra $C^*(G)$. It is a completion of the $*$-algebra $C_c(G)$ of continuous functions with compact support.

Proposition

Let $c : G \to \mathbb{R}$ be a continuous cocycle. Then the formula

$$\sigma_t^c(f)(\gamma) = e^{itc(\gamma)} f(\gamma), \qquad f \in C_c(G)$$

defines a strongly continuous one-parameter automorphism group σ^c of $C^*(G)$.

Diagonal automorphism groups

We can now give the following definition.

Definition

A strongly continuous one-parameter automorphism group σ of a C*-algebra A is called diagonal if there exists a locally compact groupoid with Haar system G and a continuous cocycle $c : G \to \mathbb{R}$ such that (A, σ) is isomorphic to $(C^*(G), \sigma^c)$.

Quasi-invariant measures

Definition

Let G be a locally compact groupoid with Haar system λ. A measure μ on $G^{(0)}$ is called quasi-invariant if the measures $\mu \circ \lambda$ and its inverse $\mu \circ \lambda^{-1}$ are equivalent. We denote by $D = d(\mu \circ \lambda)/d(\mu \circ \lambda^{-1})$ the Radon-Nikodym derivative.

Proposition

Let μ be a quasi-invariant measure. Then its Radon-Nikodym derivative D is a $\mathbb{R}_+^*$-valued cocycle.

KMS measures

Definition

Let G be a locally compact groupoid with Haar system, $c : G \to \mathbb{R}$ a continuous cocycle and $\beta \in \mathbb{R}$. One says that a measure μ on $G^{(0)}$ is (c, β)-KMS if it is quasi-invariant with R-N derivative $e^{-\beta c}$.

The above definition is essentially the same as the definition of Gibbs measures given by Capocaccia in the framework of statistical mechanics in 1976. It also agrees with the Dobrushin-Lanford-Ruelle definition.

Neshveyev's theorem

Theorem (Neshveyev12)

Let G and c as above. Assume that G is étale and that $G^{(0)}$ is compact. Then,

1. *Given a (c,β)-KMS probability measure μ on $G^{(0)}$ and a measurable family of states φ_x on the subgroups $G_x^x \cap c^{-1}(0)$ such that $\gamma\varphi_{s(\gamma)}\gamma^{-1} = \varphi_{r(\gamma)}$ for all $\gamma \in G$, the formula*

$$\varphi(f) = \int \varphi_x(f)d\mu(x), \qquad f \in C_c(G)$$

   defines a (σ^c,β)-KMS state.

2. *All (σ^c,β)-KMS states have the above form.*

Comments

In his recent thesis, J. Christensen extends this theorem to the case when $G^{(0)}$ is no longer compact. He has then to consider KMS weights rather than KMS states.

I suspect that a version of this theorem holds for non-étale locally compact groupoids with Haar system.This requires to consider weights, which is always a delicate business.

The Renault-Deaconu groupoid

Given a compact space X, two open subsets U, V and a surjective local homeomorphism $T : U \to V$, we build the following semi-direct product groupoid:

$$G(X,T) = \{(x, m-n, y) : x, y \in X;\ m, n \in \mathbb{N} \text{ et } T^m x = T^n y\}$$

It is implicit in this definition that x [resp. y] belongs to the domain of T^m [resp. T^n]. It is a locally compact étale Hausdorff groupoid.

Two basic examples.

- The one-sided shift on $\prod_1^\infty \{0,1\}$.
- The map $z \mapsto z^2$ on the circle.

Quasi-product cocycles

Given $\phi \in C(U, \mathbb{R})$, we define the cocycle $c : G(X, T) \to \mathbb{R}$ by

$$c_\phi(x, m-n, y) = \sum_{k=0}^{m-1} \phi(T^k x) - \sum_{l=0}^{n-1} \phi(T^l y)$$

Similarly, given $\psi \in C(U, \mathbb{R}_+^*)$, we define the cocycle $D_\psi : G(X, T) \to \mathbb{R}_+^*$ by

$$D_\psi(x, m-n, y) = \frac{\psi(x)\psi(Tx)\ldots\psi(T^{m-1}x)}{\psi(y)\psi(Ty)\ldots\psi(T^{n-1}y)}$$

and the transfer operator $L_\psi : C_c(U) \to C_c(X)$ by

$$L_\psi f(y) = \sum_{x \in T^{-1}(\{y\})} \psi(x) f(x).$$

Conformal measures

In the thermodynamical formalism, our KMS measures are called conformal measures. They can be described as Perron-Frobenius eigenfunctions of the dual of the transfer operator.

Lemma

A probability measure μ on X is quasi-invariant with R-N derivative D_ψ iff $L^*_\psi \mu | V = \mu_{|U}$.

Corollary

Given $\varphi \in C(X, \mathbb{R})$ and $\beta \in \mathbb{R}$, a probability measure μ on X is (c_φ, β)-KMS iff $L^*_{e^{-\beta\varphi}} \mu | V = \mu_{|U}$.

Cuntz-Krieger and graph algebras

The above formalism applies to Cuntz-Krieger and graph algebras. For example, let (X, T) be the one-sided subshift of finite type given by the transition matrix M and let c be the gauge cocycle. When M is primitive, the gauge group has a single KMS state occuring at inverse temperature $\beta = \log \lambda$, where λ is the Perron-Frobenius eigenvalue of M.

Exel-Laca algebras

Exel and Laca have defined in 1999 the Cuntz-Krieger algebra of an infinite matrix $M : I \times I \to \{0, 1\}$ where the index set I is infinite countable. For convenience, we introduce the oriented graph (V, E) where the set of vertices is $V = I$ and the arrows are (i, j) where $M(i, j) = 1$. When M is not row-finite, the space X_∞ of infinite paths is not locally compact. Let us define $J(j)$ as the set of arrows (i, j) and $\mathcal{J}$ as the set of limit points of $J(j)$ as $j \to \infty$.

Definition

A terminal path is

- either an infinite path $i_0 i_1 i_2 \dots$
- or a controlled finite path $(i_0 i_1 i_2 \dots i_n ; J)$ where $J \in \mathcal{J}$ and $i_n \in J$;
- or an empty path $(\emptyset; J)$ where $J \in \mathcal{J}$.

Exel-Laca algebra as a groupoid C*-algebra

Proposition (R 99)

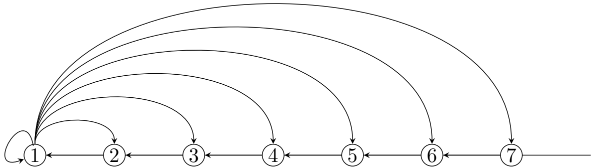
1. *the set of terminal paths* $X = X_\infty \sqcup X_{\mathrm{f}}$ *admits a natural locally compact topology;*
2. *the shift* $T : U \to X$, *where* $U = X \setminus \{(\emptyset; J), J \in \mathcal{J}\}$, *is a local homeomorphism.*

Theorem (R 99)

Exel-Laca algebra is the groupoid C-algebra* $C^*(G(X, T))$.

The renewal shift

Bissacot, Exel, Frausino, Raszeja have recently revisited the theory of conformal measures on countable Markov chains, using the framework of Exel-Laca algebras. Their pet example is the famous renewal shift.



Conformal measures on the renewal shift

Let us first determine the terminal path space of the renewal shift. Since $J(j) = \{1, j+1\}$, the only limit point is the set $\{1\}$. The controlled finite paths are the finite paths which end by 1. We have $X = X_\infty \sqcup X_{\mathrm{f}}$ and $U = X \setminus \{(\emptyset; \{1\})\}$.

The above authors prove the existence of conformal measures which live on X_{f}, hence which do not appear in the classical theory.

Theorem (Bissacot, Exel, Frausino, Raszeja 18)

Consider a potential $\varphi : U \to \mathbb{R}$ of the form $\varphi(i_1 i_2 \ldots) = f(i_1)$ where $f : I \to \mathbb{R}$ admits a strictly positive infimum $M > 0$. Then

1. *if $\beta > \log 2/M$, there exists a unique (c_φ, β)-KMS measure which vanishes on X_∞;*
2. *if $\beta < \log 2/M$, there exists no (c_φ, β)-KMS measure which vanish on X_∞.*

References

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The End

Thank you for your attention!