Subnormality of operators of class Q

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The Cauchy dual operator

- The Cauchy dual (operator) T' of a left-invertible operator $T \in \boldsymbol{B}(\mathcal{H})$ is defined by

$$T' = T(T^*T)^{-1}.$$

- If T is left-invertible, then T' is again left-invertible and

$$(T')' = T,$$
$$T^*T' = I \quad \text{and} \quad T'^*T = I.$$

- This notion has been introduced and studied by Shimorin in the context of the wandering subspace problem for Bergman-type operators (2001).

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Complete hyperexpansivity; m-isometries

- Given $m \geqslant 1$, we say that an operator $T \in \boldsymbol{B}(\mathcal{H})$ is an *m-isometry* if $B_m(T) = 0$, where

$$B_m(T) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*k} T^k.$$

- We say that T is:
  • *completely hyperexpansive* if $B_m(T) \leqslant 0$ for all $m \geqslant 1$.
  • *2-hyperexpansive* if $B_2(T) \leqslant 0$.
- • 2-hyperexpansive operator $\rightsquigarrow$ Richter (1988)
  • m-isometric operator $\rightsquigarrow$ Agler (1990)
  • completely hyperexpansive operator $\rightsquigarrow$ Athavale (1996)
- • a 2-isometry is m-isometric for every $m \geqslant 2$, and thus it is completely hyperexpansive,
  • a 2-hyperexpansive (e.g. 2-isometric) operator is left-invertible and its Cauchy dual T' is a contraction.

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- An operator $T \in \boldsymbol{B}(\mathcal{H})$ is said to be:
  • hyponormal if $TT^* \leqslant T^*T$ (Halmos 1950),
  • subnormal if there exist a Hilbert space $\mathcal{K}$ and a normal operator $N \in \boldsymbol{B}(\mathcal{K})$, i.e. $N^*N = NN^*$, such that $\mathcal{H} \subseteq \mathcal{K}$ and $Th = Nh$ for all $h \in \mathcal{H}$ (Halmos 1950),
  • quasinormal if $TTT^* = TT^*T$ (A. Brown 1953).
- quasinormal operators are subnormal and subnormal operators are hyponormal, but not reversely (if $\mathcal{H}$ is infinite dimensional).

Hyponormal, quasinormal and subnormal operators

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The Cauchy dual subnormality problem

- The map $T \longmapsto T'$ sends
  ♣ 2-hyperexpansive operators into hyponormal contractions (Shimorin 2002),
  ♣ completely hyperexpansive unilateral weighted shifts into subnormal contractions (Athavale 1996).
- This leads to the Cauchy dual subnormality problem originally posed by Chavan (2007):
- Is the Cauchy dual of a completely hyperexpansive operator a subnormal contraction?
- The Cuchy dual operator of a 2-hyperexpansive operator is power hyponormal contractions (Chavan 2013).
- The answer is NO even for 2-isometries (Anand, Chavan, Jablonski, JS 2017).

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Affirmative solutions

- The following question was addressed in (ACJS 2017):
  • find subclasses of the class of 2-isometries for which the Cauchy dual subnormality problem has an affirmative solution.
- It was proved in (ACJS 2017) that this is the case for:
  • 2-isometries satisfying the kernel condition

$$T^*T(\ker(T^*)) \subseteq \ker(T^*),$$

  • the so-called quasi-Brownian isometries.
- A recent generalization: in the class of quasi-Brownian isometries the map $T \mapsto T'$ sends bijectively hyperexpansive operators onto subnormal contractions (Badea, Suciu 2018).
- Quasi-Brownian isometries $=$ $\triangle_T$-regular 2-isometries (Majdak, Mbekhta, Suciu 2016) generalize Brownian isometries studied by Agler and Stankus (1995-1996).

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A block matrix representation

Theorem

If $T \in \boldsymbol{B}(\mathcal{H})$, then TFAE:

(i) *T is a quasi-Brownian isometry (resp., Brownian isometry),*

(ii) *T has a block matrix form*

$$T = \begin{bmatrix} V & E \\ 0 & U \end{bmatrix} \tag{1}$$

with respect to an orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $V \in \boldsymbol{B}(\mathcal{H}_1)$, $E \in \boldsymbol{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $U \in \boldsymbol{B}(\mathcal{H}_2)$ are such that

$$V \text{ isometry, } V^*E = 0, \ U \text{ isometry, } UE^*E = E^*EU \tag{2}$$

$$(\text{resp., } V \text{ isometry, } V^*E = 0, \ U \text{ unitary, } UE^*E = E^*EU), \tag{3}$$

(iii) *T is either an isometry or it has the block matrix form (1) with respect to a nontrivial orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $V \in \boldsymbol{B}(\mathcal{H}_1)$, $E \in \boldsymbol{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $U \in \boldsymbol{B}(\mathcal{H}_2)$ satisfy (2) (resp. (3)) and $\ker E = \{0\}$.*

Aims of the talk

- – This leads to a question why this phenomenon can happen.
  – We will attempt to answer this by indicating and testing a certain class of operators closed for the operation of taking the Cauchy dual.
- For this, we embed the class of quasi-Brownian isometries into an essentially larger class of operators having the 2×2 block matrix representation described by (1) and (2), not requiring that U (the bottom right corner) is an isometry.
- The entry U can be replaced by a more general operator, namely by a normal, a quasinormal or a subnormal operator; $\mathcal{N}$, $\mathcal{Q}$ and $\mathcal{S}$ denote the respective classes of operators.
- The most challenging problem is to characterize subnormality and complete hyperexpansivity within these classes. In my talk I will concentrate on the class $\mathcal{Q}$.

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The class $\mathcal{Q}$

Definition

We say that an operator $T \in \boldsymbol{B}(\mathcal{H})$ is of class $\mathcal{Q}$ if T has a block matrix form

$$T = \begin{bmatrix} V & E \\ 0 & Q \end{bmatrix}$$

with respect to a nontrivial orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $V \in \boldsymbol{B}(\mathcal{H}_1)$, $E \in \boldsymbol{B}(\mathcal{H}_2, \mathcal{H}_1)$ and $Q \in \boldsymbol{B}(\mathcal{H}_2)$ satisfy the conditions

$$V \text{ isometry}, \; V^*E = 0, \; QE^*E = E^*EQ,$$
$$Q \text{ is a quasinormal operator.}$$

If this is the case, then we write $T = \left[\begin{smallmatrix} V & E \\ 0 & Q \end{smallmatrix}\right] \in \mathcal{Q}_{\mathcal{H}_1, \mathcal{H}_2}$.

The Cauchy dual

Proposition

Suppose $T = \left[\begin{smallmatrix} V & E \\ 0 & Q \end{smallmatrix}\right] \in \mathcal{Q}_{\mathcal{H}_1,\mathcal{H}_2}$ *is left invertible. Then* $\Omega := E^*E + Q^*Q$ *is invertible in* $\boldsymbol{B}(\mathcal{H})$, $T' \in \mathcal{Q}_{\mathcal{H}_1,\mathcal{H}_2}$ *and*

$$T' = \begin{bmatrix} V & \widetilde{E} \\ 0 & \widetilde{Q} \end{bmatrix},$$

where

$$\widetilde{E} = E\Omega^{-1} \quad \text{and} \quad \widetilde{Q} = Q\Omega^{-1}.$$

Powers

Proposition

Suppose $T = \left[\begin{smallmatrix} V & E \\ 0 & Q \end{smallmatrix}\right] \in \mathcal{Q}_{\mathcal{H}_1,\mathcal{H}_2}$. *Then*

(i) $T^n = \begin{bmatrix} V^n & E_n \\ 0 & Q^n \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1,\mathcal{H}_2}$ *for any* $n \in \mathbb{Z}_+$, *where*

$$E_n = \begin{cases} 0 & \text{if } n = 0, \\ \sum_{j=1}^{n} V^{j-1} E Q^{n-j} & \text{if } n \geqslant 1, \end{cases}$$

(ii) $T^{*n} T^n = \begin{bmatrix} I & 0 \\ 0 & \Omega_n \end{bmatrix} \in \mathcal{Q}_{\mathcal{H}_1,\mathcal{H}_2}$ *for any* $n \in \mathbb{Z}_+$, *where*

$$\Omega_n = \begin{cases} I & \text{if } n = 0, \\ E^* E \left(\sum_{j=0}^{n-1} (Q^* Q)^j \right) + (Q^* Q)^n & \text{if } n \geqslant 1. \end{cases}$$

Corollary

Suppose $T = \left[\begin{smallmatrix} V & E \\ 0 & Q \end{smallmatrix}\right] \in \mathcal{Q}_{\mathcal{H}_1,\mathcal{H}_2}$. *Then* T *is an isometry if and only if*

$$|Q|^2 + |E|^2 = I, \tag{4}$$

or equivalently if and only if

$$\sigma(|Q|, |E|) \subseteq \mathbb{T}_+,$$

where $\mathbb{T}_+ := \{(x_1, x_2) \in \mathbb{R}_+^2 \colon x_1^2 + x_2^2 = 1\}$.

- A pair $(T_1, T_2) \in \boldsymbol{B}(\mathcal{H})^2$ is said to be a spherical isometry if $T_1^*T_1 + T_2^*T_2 = I$. Thus (4) means that the pair $(|Q|, |E|)$ is a spherical isometry.

Corollary

Suppose $T = \left[\begin{smallmatrix} V & E \\ 0 & Q \end{smallmatrix}\right] \in \mathcal{Q}_{\mathcal{H}_1,\mathcal{H}_2}$. *Then T is an isometry if and only if*

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$\triangle_T$-regularity

Proposition

Suppose $T = \left[\begin{smallmatrix} V & E \\ 0 & Q \end{smallmatrix}\right] \in \mathcal{Q}_{\mathcal{H}_1,\mathcal{H}_2}$. *Set* $\triangle_T = T^*T - I$ *and* $\Omega = E^*E + Q^*Q$. *Then the following conditions are equivalent*:

(i) T *is* $\triangle_T$*-regular, i.e.,* $\triangle_T \geqslant 0$ *and* $\triangle_T T = \triangle_T^{1/2} T \triangle_T^{1/2}$ *with* $\triangle_T = T^*T - I$,

(ii) $\triangle_T \geqslant 0$,

(iii) $\Omega \geqslant I$,

(iv) $\sigma(|Q|, |E|) \cap \mathbb{D}_+ = \emptyset$, *where* $\mathbb{D}_+ = \{(x_1, x_2) \in \mathbb{R}_+^2 : x_1^2 + x_2^2 < 1\}$.

Theorem

Suppose $T = \left[\begin{smallmatrix} V & E \\ 0 & Q \end{smallmatrix}\right] \in \mathcal{Q}_{\mathcal{H}_1,\mathcal{H}_2}$. *Then the following conditions are equivalent*:

(i) T *is a quasi-Brownian isometry,*

(ii) T *is a* 2*-isometry,*

(iii) $(|Q|^2 - I)(|Q|^2 + |E|^2 - I) = 0,$

(iv) $\sigma(|Q|, |E|) \subseteq \mathbb{T}_+ \cup \big(\{1\} \times \mathbb{R}_+\big),$

(v) *there exists an orthogonal decomposition* $\mathcal{H}_2 = \mathcal{H}_{2,1} \oplus \mathcal{H}_{2,2}$ *such that* $\mathcal{H}_{2,1}$ *and* $\mathcal{H}_{2,2}$ *reduce* Q *and* $|E|$, $Q|_{\mathcal{H}_{2,1}}$ *is an isometry and* $\Big(Q|_{\mathcal{H}_{2,2}}, |E||_{\mathcal{H}_{2,2}}\Big)$ *is a spherical isometry.*

Brownian-isometries

Proposition

Suppose $T = \left[\begin{smallmatrix} V & E \\ 0 & Q \end{smallmatrix}\right] \in \mathcal{Q}_{\mathcal{H}_1,\mathcal{H}_2}$. *Then the following conditions are equivalent*:

(i) T *is a Brownian isometry, i.e.,* T *is a* 2*-isometry such that* $\triangle_T \triangle_{T^*} \triangle_T = 0$,

(ii) $(|Q|^2 - I)(|Q|^2 + |E|^2 - I) = 0$

and

$(|Q^*|^2 - I)(|Q|^2 + |E|^2 - I)^2 = 0$,

(iii) *there exists an orthogonal decomposition* $\mathcal{H}_2 = \mathcal{H}_{2,1} \oplus \mathcal{H}_{2,2}$ *such that* $\mathcal{H}_{2,1}$ *and* $\mathcal{H}_{2,2}$ *reduce* Q *and* $|E|$, $Q|_{\mathcal{H}_{2,1}}$ *is an isometry,* $\left(Q|_{\mathcal{H}_{2,2}}, |E||_{\mathcal{H}_{2,2}}\right)$ *is a spherical isometry and the spaces* $\mathcal{H}_{2,1} \ominus Q(\mathcal{H}_{2,1})$ *and* $|E|(\mathcal{H}_{2,1})$ *are orthogonal.*

Subnormality

Theorem

Suppose $T = \left[\begin{smallmatrix} V & E \\ 0 & Q \end{smallmatrix}\right] \in \mathcal{Q}_{\mathcal{H}_1,\mathcal{H}_2}$. *Then the following conditions are equivalent*:

(i) T *is subnormal,*

(ii) $\sigma(|Q|,|E|) \subseteq \bar{\mathbb{D}}_+ \cup \Big((1,\infty) \times \{0\}\Big)$,

where $\sigma(|Q|,|E|)$ *stands for the Taylor spectrum of* $(|Q|,|E|)$ *and*

$$\mathbb{D}_+ = \{(x_1,x_2) \in \mathbb{R}_+^2 \colon x_1^2 + x_2^2 < 1\}.$$

Hamburger and Stieltjes moment multi-sequences

We say that a multi-sequence $\{\gamma_\alpha\}_{\alpha\in\mathbb{Z}_+^d} \subseteq \mathbb{R}$ is a Hamburger moment multi-sequence (or Hamburger moment sequence if $d = 1$) if there exists a positive Borel measure μ on $\mathbb{R}^d$, called a representing measure of $\{\gamma_\alpha\}_{\alpha\in\mathbb{Z}_+^d}$, such that

$$\gamma_\alpha = \int_{\mathbb{R}^d} x^\alpha d\mu(x), \quad \alpha \in \mathbb{Z}_+^d. \tag{5}$$

If such μ is unique, then $\{\gamma_\alpha\}_{\alpha\in\mathbb{Z}_+^d}$ is said to be determinate. If (5) holds for some positive Borel measure μ on $\mathbb{R}^d$ supported in $\mathbb{R}_+^d$, then $\{\gamma_\alpha\}_{\alpha\in\mathbb{Z}_+^d}$ is called a Stieltjes moment multi-sequence (or Stieltjes moment sequence if $d = 1$).

Lambert's characterization of subnormality

Theorem (Lambert 1976)

An operator $T \in \boldsymbol{B}(\mathcal{H})$ is subnormal if and only if for every $f \in \mathcal{H}$, the sequence $\{\|T^n f\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence, i.e., there exists a positive Borel measure μ_f on $[0, \infty)$ such that

$$\|T^n f\|^2 = \int_{[0,\infty)} t^n d\mu_f(t), \quad n = 0, 1, 2, \ldots.$$

Spectral integrals and moments

Lemma

Let $G\colon \mathfrak{B}(X) \to \boldsymbol{B}(\mathcal{H})$ be a regular Borel spectral measure on a topological Hausdorff space X such that $\operatorname{supp} G$ is compact. Suppose that for every $n \in \mathbb{Z}_+$, $\varphi_n\colon X \to \mathbb{R}$ is a continuous function. Then TFAE:

(i) $\{\int_X \varphi_n(x)\langle G(\mathrm{d}\, x)f, f\rangle\}_{n=0}^{\infty}$ *is a Stieltjes moment sequence for every $f \in \mathcal{H}$,*

(ii) $\{\varphi_n(x)\}_{n=0}^{\infty}$ *is a Stieltjes moment sequence for every $x \in \operatorname{supp} G$.*

A characterization of subnormality

Theorem

Let $G\colon \mathfrak{B}(X) \to \boldsymbol{B}(\mathcal{H})$ be a regular Borel spectral measure on a topological Hausdorff space X such that $\operatorname{supp} G$ is compact and let $T \in \boldsymbol{B}(\mathcal{H})$ be such that

$$T^{*n}T^n = \int_X \varphi_n \,\mathrm{d}\, G, \quad n \in \mathbb{Z}_+,$$

where $\varphi_n\colon X \to \mathbb{R}$, $n \in \mathbb{Z}_+$, are continuous functions. Suppose that for every $x \in X$, there exists a compactly supported complex Borel measure μ_x on $\mathbb{R}_+$ such that

$$\varphi_n(x) = \int_{\mathbb{R}_+} t^n \,\mathrm{d}\, \mu_x(t), \quad n \in \mathbb{Z}_+,\ x \in X.$$

Then T is subnormal if and only if for every $x \in \operatorname{supp} G$, μ_x is a positive measure.

Hamburger multi-sequences

Lemma

Let $d \in \mathbb{N}$, μ be a compactly supported complex Borel measure on $\mathbb{R}^d$ and

$$\gamma_\alpha = \int_{\mathbb{R}^d} x^\alpha d\mu(x), \quad \alpha \in \mathbb{Z}_+^d.$$

Then the following conditions are equivalent:

(i) $\{\gamma_\alpha\}_{\alpha \in \mathbb{R}^d}$ *is a Hamburger moment multi-sequence,*

(ii) μ *is a positive measure.*

Moreover, if (i) *holds, then* $\{\gamma_\alpha\}_{\alpha \in \mathbb{R}^d}$ *is determinate.*

Uniqueness of moments I

Lemma

Suppose $d \geqslant 1$ and μ_1 and μ_2 are compactly supported complex Borel measures on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} x^\alpha d\mu_1(x) = \int_{\mathbb{R}^d} x^\alpha d\mu_2(x), \quad \alpha \in \mathbb{Z}_+^d.$$

Then $\mu_1 = \mu_2$.

Any sequence $\{\gamma_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$ has infinitely many representing complex measures. Indeed, by [Boas 1938, Durán], there is a complex Borel measure ρ on $\mathbb{R}$ such that

$$\gamma_n = \int_{\mathbb{R}} x^n d\rho(x), \quad n \in \mathbb{Z}_+.$$

Let $\{s_n\}_{n=0}^{\infty}$ be an indeterminate Hamburger moment sequence with two distinct representing measures μ_1 and μ_2. Then $\mu := \mu_1 - \mu_2$ is a signed Borel measure on $\mathbb{R}$ such that (Stieltjes)

$$\int_{\mathbb{R}} x^n d\mu(x) = 0, \quad n \in \mathbb{Z}_+.$$

As a consequence, we have

$$\gamma_n = \int_{\mathbb{R}} x^n d(\rho + \vartheta\mu)(x), \quad n \in \mathbb{Z}_+, \vartheta \in \mathbb{C}.$$

Moreover, the mapping $\mathbb{C} \ni \vartheta \longmapsto \rho + \vartheta\mu$ is injective.

Lemma

Let for $k = 1, 2$, $\{\gamma_k(n)\}_{n=0}^{\infty}$ be a Hamburger moment sequence having a compactly supported representing measure μ_k and let $p \in \mathbb{C}[x]$ be such that

$$\gamma_1(n) = \gamma_2(n) + p(n), \quad n \in \mathbb{Z}_+. \tag{6}$$

Then p is a constant polynomial and $\mu_1 = \mu_2 + p(0)\delta_1$.

Corollary

Suppose $p \in \mathbb{C}[x]$. Then the following conditions are equivalent:

(i) $\{p(n)\}_{n=0}^{\infty}$ *is a Hamburger moment sequence,*

(ii) $\{p(n)\}_{n=0}^{\infty}$ *is a Stieltjes moment sequence,*

(iii) p *is a constant polynomial and* $p(0) \geqslant 0$.

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Corollary

Suppose $p \in \mathbb{C}[x]$. Then the following conditions are equivalent:

(i) $\{p(n)\}_{n=0}^{\infty}$ *is a Hamburger moment sequence,*

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THANK YOU!