Wold decomposition for doubly commuting isometric covariant representations

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Introduction to Wold decomposition

- Let V be an **isometry** on a Hilbert space $\mathcal{H}$, that is, $V^*V = I_{\mathcal{H}}$.
- A closed subspace $\mathcal{W} \subseteq \mathcal{H}$ is said to be **wandering subspace** for V if $V^k\mathcal{W} \perp V^l\mathcal{W}$ for all $k, l \in \mathbb{N}$ with $k \neq l$, or equivalently, if $V^m\mathcal{W} \perp \mathcal{W}$ for all $m \geq 1$.
- An isometry V on $\mathcal{H}$ is said to be a **unilateral shift** or **shift** if

$$\mathcal{H} = \bigoplus_{m \geq 0} V^m\mathcal{W},$$

for some wandering subspace $\mathcal{W}$ for V.

- For a shift V on $\mathcal{H}$ with a wandering subspace $\mathcal{W}$ we have

$$\mathcal{H} \ominus V\mathcal{H} = \bigoplus_{m\geq 0} V^m\mathcal{W} \ominus V(\bigoplus_{m\geq 0} V^m\mathcal{W})$$

$$= \bigoplus_{m\geq 0} V^m\mathcal{W} \ominus \bigoplus_{m\geq 1} V^m\mathcal{W} = \mathcal{W}.$$

- The wandering subspace of a shift is unique and is given by

$$\mathcal{W} = \mathcal{H} \ominus V\mathcal{H}.$$

Theorem (Wold, 1938)

Let V be an isometry on $\mathcal{H}$. Then $\mathcal{H}$ admits a unique decomposition $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_u$, where $\mathcal{H}_s$ and $\mathcal{H}_u$ are V-reducing subspaces of $\mathcal{H}$ and $V|_{\mathcal{H}_s}$ is a shift and $V|_{\mathcal{H}_u}$ is unitary. Moreover,

$$\mathcal{H}_s = \bigoplus_{m=0}^{\infty} V^m \mathcal{W} \qquad \text{and} \qquad \mathcal{H}_u = \bigcap_{m=0}^{\infty} V^m \mathcal{H},$$

where $\mathcal{W} = ran(I - VV^*) = kerV^*$ is the wandering subspace for V.

- Let $V = (V_1, \ldots, V_n)$ be an n-tuple ($n \geq 2$) of commuting isometries on $\mathcal{H}$. Then V is said to *doubly commute* if

$$V_i V_j^* = V_j^* V_i,$$

for all $1 \leq i < j \leq n$.

Theorem (M. Slocinski, 1980)

Let $V = (V_1, V_2)$ be a pair of doubly commuting isometries on a Hilbert space $\mathcal{H}$. Then there exists a unique decomposition

$$\mathcal{H} = \mathcal{H}_{ss} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{uu},$$

where $\mathcal{H}_{ij}$ are joint V-reducing subspace of $\mathcal{H}$ for all $i, j = s, u$. Moreover, V_1 on $\mathcal{H}_{i,j}$ is a shift if $i = s$ and unitary if $i = u$ and that V_2 is a shift if $j = s$ and unitary if $j = u$.

Notation:

Let $(T_1, \dots, T_n)$ be an n-tuple of commuting operators on a Hilbert space $\mathcal{H}$ and $1 \leq m \leq n$. Let $A = \{i_1, \dots, i_l\} \subseteq I_m$ and $1 \leq l \leq m$. We denote by T_A the $|A|$-tuple of commuting operators $(T_{i_1}, \dots, T_{i_l})$ and $\mathbb{N}^A := \{\mathbf{k} = (k_{i_1}, \dots, k_{i_l}) : k_{i_j} \in \mathbb{N}, 1 \leq j \leq l\}$. We also denote $T_{i_1}^{k_{i_1}} \cdots T_{i_l}^{k_{i_l}}$ by $T_A^{\mathbf{k}}$ for all $\mathbf{k} \in \mathbb{N}^A$.

Theorem (J. Sarkar, 2014)

Let $V = (V_1, \dots, V_n)$ be an n-tuple ($n \geq 2$) of doubly commuting isometries on $\mathcal{H}$ and $m \in \{2, \dots, n\}$. Let $I_m = \{1, 2, \dots, m\}$. Then there exists 2^m joint $(V_1, \dots, V_m)$-reducing subspaces $\{\mathcal{H}_A : A \subseteq I_m\}$ (counting the trivial subspace $\{0\}$) such that

$$\mathcal{H} = \bigoplus_{A \subseteq I_m} \mathcal{H}_A, \tag{1}$$

where for each $A \subseteq I_m$,

$$\mathcal{H}_A = \bigoplus_{\mathbf{k} \in \mathbb{N}^A} V_A^{\mathbf{k}} \Big(\bigcap_{\mathbf{j} \in \mathbb{N}^{I_m \setminus A}} V_{I_m \setminus A}^{\mathbf{j}} \mathcal{W}_A \Big). \tag{2}$$

In particular, there exists 2^n orthogonal joint V-reducing subspaces $\{\mathcal{H}_A : A \subseteq I_n\}$ such that

$$\mathcal{H} = \sum_{A \subseteq I_n} \oplus \mathcal{H}_A,$$

and for each $A \subseteq I_n$ and $\mathcal{H}_A \neq \{0\}$, $V_i|_{\mathcal{H}_A}$ is a shift if $i \in A$ and unitary if $i \in I_n \setminus A$ for all $i = 1, \ldots, n$. Moreover, the above decomposition is unique, in the sense that

$$\mathcal{H}_A = \bigoplus_{\mathbf{k} \in \mathbb{N}^A} V_A^{\mathbf{k}} \Big(\bigcap_{\mathbf{j} \in \mathbb{N}^{I_n \setminus A}} V_{I_n \setminus A}^{\mathbf{j}} \mathcal{W}_A \Big),$$

for all $A \subseteq I_n$.

- This decomposition is stronger in the sense that the orthogonal decomposition works for any $m \in \{2, \ldots, n\}$ with $(2 < n)$.

Definition

Let $\mathcal{M}$ be a C^*-algebra and E be a vector space which is a right $\mathcal{M}$-module satisfying $\alpha(xm) = (\alpha x)m = x(\alpha m)$ for $x \in E, m \in \mathcal{M}, \alpha \in \mathbb{C}$. The module E is called an **(right) inner-product $\mathcal{M}$-module**
if there exists a map $\langle \cdot, \cdot \rangle : E \times E \to \mathcal{M}$ satisfying :

(i) $\langle x, x \rangle \geq 0$ for $x \in E$ and $\langle x, x \rangle = 0$ only if $x = 0$,

(ii) $\langle x, ym \rangle = \langle x, y \rangle m$ for $x, y \in E$ and for $m \in \mathcal{M}$,

(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for $x, y \in E$,

(iv) $\langle x, \mu y + \nu z \rangle = \mu \langle x, y \rangle + \nu \langle x, z \rangle$ for $x, y, z \in E$ and for $\mu, \nu \in \mathbb{C}$.

Definition

A **(right) Hilbert $\mathcal{M}$-module** is an inner-product $\mathcal{M}$-module E which is complete w.r.t. $\|x\| := \|\langle x, x \rangle\|^{1/2}$ for $x \in E$.

- Let E be a Hilbert $\mathcal{M}$-modules. A map $T : E \to E$ is called **adjointable** if there exists a map $S : E \to E$ such that

$$\langle T(x), y\rangle = \langle x, S(y)\rangle \text{ for all } x, y \in E.$$

Notation: $\mathcal{L}(E)$.

- The module E is said to be a C^*-**correspondence over** $\mathcal{M}$ if it has a left $\mathcal{M}$-module structure induced by a non-zero $*$-homomorphism $\phi : \mathcal{M} \to \mathcal{L}(E)$ in the sense $a\xi := \phi(a)\xi \qquad (a \in \mathcal{M}, \xi \in E).$
- If F is another C^*-correspondence over $\mathcal{M}$, then **tensor product** $F \bigotimes_\phi E$ satisfy the following properties: for all $\zeta_1, \zeta_2 \in F$, $\xi_1, \xi_2 \in E$ and $a \in \mathcal{M}$

$$(\zeta_1 a) \otimes \xi_1 = \zeta_1 \otimes \phi(a)\xi_1,$$

$$\langle \zeta_1 \otimes \xi_1, \zeta_2 \otimes \xi_2\rangle = \langle \xi_1, \phi(\langle \zeta_1, \zeta_2\rangle)\xi_2\rangle.$$

Definition

Let E be a C^*-correspondence over $\mathcal{M}$ and $\mathcal{H}$ be a Hilbert space. Assume $\sigma : \mathcal{M} \to B(\mathcal{H})$ to be a representation and $T : E \to B(\mathcal{H})$ to be a linear map. The tuple (T, σ) is called a **covariant representation** of E on $\mathcal{H}$ if

$$T(m\xi m') = \sigma(m)T(\xi)\sigma(m') \qquad (\xi \in E, m, m' \in \mathcal{M}). \tag{3}$$

The covariant representation is called **completely contractive** if T is completely contractive. The covariant representation (T, σ) is called **isometric** if

$$T(\xi_1)^* T(\xi_2) = \sigma(\langle \xi_1, \xi_2 \rangle) \qquad (\xi_1, \xi_2 \in E).$$

Lemma (Muhly and Solel, 1998)

The map $(T, \sigma) \mapsto \widetilde{T}$ *provides a bijection between the collection of all completely contractive, covariant representations* (T, σ) *of* E *on* $\mathcal{H}$ *and the collection of all contractive linear maps* $\widetilde{T}: \ E \bigotimes_{\sigma} \mathcal{H} \to \mathcal{H}$ *defined by*

$$\widetilde{T}(\xi \otimes h) := T(\xi)h \qquad (\xi \in E, h \in \mathcal{H}),$$

and such that $\widetilde{T}(\phi(a) \otimes I_{\mathcal{H}}) = \sigma(a)\widetilde{T}$, $a \in \mathcal{M}$. *Moreover,* $\widetilde{T}$ *is isometry if and only if* (T, σ) *is isometric.*

Let E be a C^*-correspondence over a C^*-algebra $\mathcal{M}$. Then for each $n \in \mathbb{N}$, $E^{\otimes n} := E \otimes_\phi \cdots \otimes_\phi E$ (n times) is the C^*-correspondence over the C^*-algebra $\mathcal{M}$, where the left action of $\mathcal{M}$ on $E^{\otimes n}$ is given by

$$\phi^n(a)(\xi_1 \otimes \cdots \otimes \xi_n) := \phi(a)\xi_1 \otimes \cdots \otimes \xi_n.$$

Denote $E^{\otimes 0} := \mathcal{M}$. The Fock space $\mathcal{F}(E) := \bigoplus_{n \geq 0} E^{\otimes n}$ is the C^* correspondence over a C^*-algebra $\mathcal{M}$, with left action of $\mathcal{M}$ on $\mathcal{F}(E)$ is given by $\phi_\infty : \mathcal{M} \longrightarrow L(\mathcal{F}(E))$ where

$$\phi_\infty(a)(\oplus_{n \geq 0}\xi_n) := \oplus_{n \geq 0} a \cdot \xi_n, \;\; \xi_n \in E^{\otimes n}.$$

Let $\xi \in E$, we define the creation operator T_ξ on $\mathcal{F}(E)$ by

$$T_\xi(\eta) := \xi \otimes \eta, \;\; \eta \in E^{\otimes n}.$$

Let π be a representation of $\mathcal{M}$ on the Hilbert space $\mathcal{H}$. The isometric covariant representation (ρ, S) of E on the Hilbert space $\mathcal{F}(E) \otimes_\pi \mathcal{H}$ defined by

$$\begin{aligned} \rho(a) &:= \phi_\infty(a) \otimes I_{\mathcal{H}} \ , a \in \mathcal{M} \\ S(\xi) &:= T_\xi \otimes I_{\mathcal{H}}, \ \ \xi \in E. \end{aligned}$$

is called an **induced representation** (induced by π).

Definition (L. Helmer, 2016)

Let E be a C^*-correspondence over a C^*-algebra $\mathcal{M}$. Let (σ, V) be an isometric covariant representation of E on a Hilbert space $\mathcal{H}$. For a closed $\sigma(\mathcal{M})$-invariant subspace $\mathcal{W}$, we define

$$\mathfrak{L}^n(\mathcal{W}) := \bigvee\{V(\xi_1)V(\xi_2)\dots V(\xi_n)h \,:\, \xi_i \in E, h \in \mathcal{W}\},$$

for $n \in \mathbb{N}$ and $\mathfrak{L}^0(\mathcal{W}) := \mathcal{W}$. Then $\mathcal{W}$ is called **wandering** for (σ, V), if the subspaces $\mathfrak{L}^n(\mathcal{W})$, $n \in \mathbb{N}_0$ are mutually orthogonal.

Theorem (Muhly-Solel, 1999)

Let (σ, V) be an isometric covariant representation of E on a Hilbert space $\mathcal{H}$. Then the representation (σ, V) decomposes into a direct sum $(\sigma_1, V_1) \bigoplus (\sigma_2, V_2)$ on $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_2$ where $(\sigma_1, V_1) = (\sigma, V)|_{\mathcal{H}_1}$ is an induced covariant representation and $(\sigma_2, V_2) = (\sigma, V)|_{\mathcal{H}_2}$ is fully coisometric. The above decomposition is unique in the sense that if $\mathcal{K}$ reduces (σ, V) , and if the restriction $(\sigma, V)|_{\mathcal{K}}$ is induced(resp. fully coisometric), then $\mathcal{K} \subset \mathcal{H}_1$(resp. $\mathcal{K} \subset \mathcal{H}_2$). Moreover, $\mathcal{H}_2 := \bigoplus_{k \geq 0} \mathfrak{L}^k(\mathcal{W})$, and hence

$$\mathcal{H}_1 := \left(\bigoplus_{k \geq 0} \mathfrak{L}^k(\mathcal{W}) \right)^{\perp} = \bigcap_{k=0}^{\infty} (\mathfrak{L}^k(\mathcal{H})).$$

Let $k \in \mathbb{N}$ and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We require *product system* $\mathbb{E}$ *of C^*-correspondences over* $\mathbb{N}_0^k$[Fowler, 2002]: Consider $\mathbb{E}$ to be a family of k C^*-correspondences $\{E_1, \dots, E_k\}$ together with the unitary isomorphisms $t_{i,j} : E_i \otimes E_j \to E_j \otimes E_i$ $(i > j)$. Thus we identify for all $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}_0^k$ the correspondence $\mathbb{E}(\mathbf{n})$ with $E_1^{\otimes^{n_1}} \otimes \cdots \otimes E_k^{\otimes^{n_k}}$. Indeed, we use $t_{i,i} = \mathrm{id}_{E_i \otimes E_i}$, $t_{i,j} = t_{j,i}^{-1}$ for $i < j$.

Definition

Assume $\mathbb{E}$ to be a product system over $\mathbb{N}_0^k$. By a **completely contractive covariant representation** of $\mathbb{E}$ on a Hilbert space $\mathcal{H}$ we mean a tuple $(\sigma, T^{(1)}, \ldots, T^{(k)})$, where $(\sigma, \mathcal{H})$ is a representation of $\mathcal{M}$, and $T^{(i)} : E_i \to B(\mathcal{H})$ are linear completely contractive maps satisfying

$$T^{(i)}(a\xi_i b) = \sigma(a) T^{(i)}(\xi_i)\sigma(b), \quad a, b \in \mathcal{M}, \xi_i \in E_i,$$

as well as $\widetilde{T}^{(i)}(I_{E_i} \otimes \widetilde{T}^{(j)}) = \widetilde{T}^{(j)}(I_{E_j} \otimes \widetilde{T}^{(i)})(t_{i,j} \otimes I_{\mathcal{H}})$ with $i, j \in \{1, \ldots, k\}$.

Moreover, the representation is called **isometric** if each $(\sigma, T^{(i)})$ is isometric as a representation of E_i, and **fully coisometric** if each $(\sigma, T^{(i)})$ is fully coisometric.

Definition

A representation $(\sigma, T^{(1)}, \ldots, T^{(k)})$ of $\mathbb{E}$ on a Hilbert space $\mathcal{H}$ is called **doubly commuting** if for each $i, j \in \{1, \ldots, k\}$, $i \neq j$ implies

$$\widetilde{T}^{(j)^*} \widetilde{T}^{(i)} = (I_{E_j} \otimes \widetilde{T}^{(i)})(t_{i,j} \otimes I_{\mathcal{H}})(I_{E_i} \otimes \widetilde{T}^{(j)^*}). \tag{4}$$

Definition

Let $\mathcal{K}$ be a closed subspace of a Hilbert space $\mathcal{H}$. The subspace $\mathcal{K}$ is called **reducing** for a doubly commuting representation $(\sigma, T^{(1)}, \ldots, T^{(k)})$ on $\mathcal{H}$, if it reduces $\sigma(\mathcal{M})$ (this means that the projection onto $\mathcal{K}$, will be denoted by $P_{\mathcal{K}}$, lies in $\sigma(\mathcal{M})'$), and both $\mathcal{K}, \mathcal{K}^{\perp}$ are left invariant by each operator $T^{(i)}(\xi_i)$ for $\xi_i \in E_i$, $i \in \{1, \ldots, k\}$. Then it is evident that the 'restriction' of this representation provides a new representation of $\mathbb{E}$ on $\mathcal{K}$, which is called a **summand** of $(\sigma, T^{(1)}, \ldots, T^{(k)})$ and will be denoted by $(\sigma, T^{(1)}, \ldots, T^{(k)})|_{\mathcal{K}}$.

Remark

To check $\mathcal{K}$ reduces $(\sigma, T^{(j)})$, it is enough to check $\mathcal{K}$ reduces $\sigma(\mathcal{M})$, and $P_{\mathcal{K}}$ commutes with $\widetilde{T}^{(j)}\widetilde{T}^{(j)^*}$.

For a closed subspace $\mathcal{K}$, we use notation $\mathfrak{L}^i_l(\mathcal{K})$ for the closed subspace generated by

$$\{T^{(i)}(\xi_1)\cdots T^{(i)}(\xi_l)k : \xi_1, \dots \xi_l \in E_i, k \in \mathcal{K}\}.$$

When $l = 1$, we denote it by $\mathfrak{L}^i(\mathcal{K})$.
For $\mathbf{n} = (n_1, \cdots, n_k) \in \mathbb{N}_0^k$, we define $\widetilde{T}_{\mathbf{n}} : \mathbb{E}(\mathbf{n}) \otimes_\sigma \mathcal{H} \longrightarrow \mathcal{H}$ by

$$\widetilde{T}_{\mathbf{n}} := \widetilde{T}^{(1)}_{n_1}\left(I_{E_1^{\otimes n_1}} \otimes \widetilde{T}^{(2)}_{n_2}\right)\cdots\left(I_{E_1^{\otimes n_1}\otimes\cdots\otimes E_{k-1}^{\otimes n_{k-1}}} \otimes \widetilde{T}^{(k)}_{n_k}\right).$$

Let $A = \{i_1, \cdots i_p\} \subset \{1, 2, \cdots, k\}$, denote $\mathbb{N}_0^A := \{\mathbf{m} = (m_{i_1}, \cdots m_{i_p}) \ : \ m_{i_j} \in \mathbb{N}_0, 1 \leq j \leq p\}$. Let $\mathbf{m} = (m_{i_1}, \cdots m_{i_p}) \in \mathbb{N}_0^A$, define $\widetilde{T}^A_{\mathbf{m}} : \mathbb{E}(\mathbf{m}) \otimes_\sigma \mathcal{H} \longrightarrow \mathcal{H}$ by

$$\widetilde{T}^A_{\mathbf{m}} = \widetilde{T}^{(i_1)}_{m_{i_1}}\left(I_{E_{i_1}^{\otimes m_{i_1}}} \otimes \widetilde{T}^{(i_2)}_{m_{i_2}}\right)\cdots\left(I_{E_{i_1}^{\otimes m_{i_1}}\otimes\cdots\otimes E_{i_{p-1}}^{\otimes m_{i_{p-1}}}} \otimes \widetilde{T}^{(i_p)}_{m_{i_p}}\right).$$

Moreover, for a given closed subspace $\mathcal{K}$, we use symbol

$$\mathfrak{L}^A_{\mathbf{m}}(\mathcal{K}) := \bigvee\{T^{(i_1)}_{m_{i_1}}(\eta_{i_1})\cdots T^{(i_p)}_{m_{i_p}}(\eta_{i_p})h : \eta_{i_j} \in E_{i_j}^{\otimes m_{i_j}}, 1 \leq j \leq p, h \in \mathcal{K}\}.$$

Clearly $\mathfrak{L}^A_{\mathbf{m}}(\mathcal{K}) = \widetilde{T}^A_{\mathbf{m}}(\mathbb{E}(\mathbf{m}) \otimes_\sigma \mathcal{K})$.

Main result

Theorem (H.-Shankar V.)

Let $\mathbb{E}$ be a product system of C^*-correspondences over $\mathbb{N}_0^k$. Let $(\sigma, T^{(1)}, \dots, T^{(k)})$ be a doubly commuting isometric, covariant representation of $\mathbb{E}$ on a Hilbert space $\mathcal{H}$. Then for $2 \leq m \leq k$, there exists 2^m $(\sigma, T^{(1)}, \dots, T^{(m)})$-reducing subspaces $\{\mathcal{H}_{\mathcal{A}} \,:\, A \subseteq I_m\}$ such that

$$\mathcal{H} := \bigoplus_{A \subseteq I_m} \mathcal{H}_A,$$

where

$$\mathcal{H}_A = \bigoplus_{\mathbf{n} \in \mathbb{N}_0^A} \mathfrak{L}_{\mathbf{n}}^A \left(\bigcap_{\mathbf{j} \in \mathbb{N}_0^{I_m \setminus A}} \mathfrak{L}_{\mathbf{j}}^{I_m \setminus A}(\mathcal{W}_A) \right).$$

Corollary (Theorem 2.4, Skalski-Zacharias, 2008)

In particular, there exist 2^k *orthogonal* $(\sigma, T^{(1)}, \ldots, T^{(k)})$*-reducing subspaces* $\{\mathcal{H}_A : A \subset I_k\}$ *such that*

$$\mathcal{H} := \bigoplus_{A \subseteq I_k} \mathcal{H}_A,$$

and for each $A \subset I_k$ *and* $\mathcal{H}_A \neq \{0\}$; $(\sigma, T^{(i)})|_{\mathcal{H}_A}$ *is an induced representation whenever* $i \in A$ *and* $(\sigma, T^{(i)})|_{\mathcal{H}_A}$ *is fully coisometric whenever* $i \in I_n \setminus A$*. Moreover, the above decomposition is unique.*

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THANK YOU.