Quantum Symmetry of classical spaces

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Some references (not exhaustive)

1. D. Goswami: Non-existence of genuine (compact) quantum symmetries of compact, connected smooth manifolds, arXiv:1805.05765.

2. D Goswami and S Joardar: Nonexistnce of faithful isometric actions on compact connected smooth manifolds, Geom. Funct. Anal. 28 (2018), no. 1, 146–178

3. D Goswami and J Bhowmick: Quantum isometry groups, Springer (2017).

4. D Goswami: Existence and examples of quantum isometry groups for a class of compact metric spaces. Adv. Math. **280** (2015), 340–359

5. D. Goswami: Quantum group of isometries in classical and noncommutative geometry. Comm. Math. Phys. **285** (2009), 141–160.

contents

It is interesting to investigate quantum group symmetry of classical spaces. In this context, one naturally asks the following:

Can there be a non-classical symmetry by a genuine quantum group of the classical system?

Here, a genuine quantum group means that the underlying algebra structure is not commutative, as we adopt the convention of symmetry given by a co-action. One can also ask a dual question replacing co-action by action where genuine should correspond to noncommnutative co-algebra structure. We address these issues in the analytic framework of C^* algebraic compact quantum groups.

Quick review of basic concepts

Definition

A **compact quantum group** *(CQG for short) a la Woronowicz is a pair* $(\mathcal{A}, \Delta)$ *where* $\mathcal{A}$ *is a unital* C^**-algebra,* Δ *is a coassociative comultiplication, i.e. a unital* C^**-homomorphism from* $\mathcal{A}$ *to* $\mathcal{A} \otimes \mathcal{A}$ *(minimal tensor product) satisfying* $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$*, and linear span of each of the sets* $\{(b \otimes 1)\Delta(c) \colon b, c \in \mathcal{A}\}$ *and* $\{(1 \otimes b)\Delta(c) \colon b, c \in \mathcal{A}\}$ *is dense in* $\mathcal{A} \otimes \mathcal{A}$*.*

Definition

We say that a CQG $(\mathcal{A}, \Delta)$ *(co)-acts on a (unital)* C^**-algebra* $\mathcal{C}$ *if there is a unital* $*$*-homomorphism* $\alpha : \mathcal{C} \to \mathcal{C} \otimes \mathcal{A}$ *such that* $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$*, and the linear span of* $\alpha(\mathcal{C})(1 \otimes \mathcal{A})$ *is norm-dense in* $\mathcal{C} \otimes \mathcal{A}$*.*

- Every CQG $\mathcal{A}$ contains a unital dense $*$-subalgebra $\mathcal{A}_0$ and maps $\kappa : \mathcal{A}_0 \to \mathcal{A}_0$, $\epsilon : \mathcal{A}_0 \to \mathbb{C}$ such that $\mathcal{A}_0$ becomes a Hopf algebra with κ, ϵ as the antipode and counit respectively. Moreover, it is a Hopf $*$-algebra , i.e. ϵ is $*$-homomorphism and $(\kappa \circ *)^2 = \mathrm{id}$.
- There is an analogue of Haar measure, namely a (unique) positive linear functional h, called the Haar state, on $\mathcal{A}$ such that $h(1) = 1$ and $(h \otimes \mathrm{id})(\Delta(a)) = (\mathrm{id} \otimes h)(\Delta(a)) = h(a)1$ for all $a \in \mathcal{A}$. Moreover, there is an exact analogue of unitary co-representation and Peter-Weyl theory.

- Given an action α of a CQG $\mathcal{A}$ on a unital C^* algebra $\mathcal{C}$, we can find a norm-dense unital $*$-subalgebra $\mathcal{C}_0$ of $\mathcal{C}$ such that α restricts to an algebraic co-action of the Hopf algebra $\mathcal{A}_0$ on $\mathcal{C}_0$.
- In Woronowicz theory, it is customary to drop 'co', and call the above co-action simply 'action' of the CQG on the C^* algebra. However, to avoid confusion, let's say that $\mathcal{A}$ acts on a space X, or, α is an action on X to mean α is a co-action on $C(X)$ in the sense of the above definition.
- A co-action α on $\mathcal{C}$ is called faithful if the $*$-subalgebra generated by $\{(\omega \otimes \mathrm{id})(\alpha(b))\}$, where $b \in \mathcal{C}$ and ω varying over the set of bounded linear functionals on $\mathcal{C}$, is dense in $\mathcal{A}$.

Genuine Quantum group actions on classical spaces

- Wang's quantum permutation group $\mathcal{S}_n^+$ acts faithfully on the finite set of cardinality n. In a similar way, one can get its action on smooth manifolds with n components, so it is a genuine CQG action for $n \geq 4$.
- There can be faithful action of $\mathcal{S}_n^+$ on more interesting, compact, connected sets (due to H. Hwang) formed by the topological join of n copies of a compact connected set, say the unit interval, gluing them at a common point. The action 'quantum permutes' these copies in the natural sense.
- There are geunine (i.e. not commutative as a C^* algebra) compact quantum group actions on connected algebraic varieties (non-smooth!) as well: Etingof and Walton give an example of $C^*(S_3)$ action on the variety $\{x, y \in \mathbb{R} : \; xy = 0\}$.

- Also, there are smooth (to be defined later) faithful actions by Hopf algebras associated with genuine **locally compact** quantum groups on noncompact, smooth, connected manifolds (e.g. $\mathbb{R}$) and even (at least algebraically) on compact connected smooth manifolds as well.
- To summarize, there are genuine quantum actions on classical connected spaces when either the space is non-smooth or the quantum group is of non-compact type, but if we demand the space to be compact and smooth, the quantum group to be compact and the action to be smooth in a natural sense at the same time, there seems to be a problem.
- This leads to the conjecture made by the speaker : **There does not exist a faithful smooth (to be defined) action by a genuine compact quantum group on a smooth, compact, connected manifold**

Isometric actions

For a smooth action by a compact group, we can use the averaging technique to get a Riemannian metric for which the group action becomes isometric. Motivated by this, one can formulate a notion of isometry for CQG action. Now, a smooth map γ on M is a Riemannian isometry if and only if the induced map $f \mapsto f \circ \gamma$ on $C^\infty(M)$ commutes with the Laplacian $\mathcal{L} = -d^*d$. Thus, it makes sense to call a CQG action α on M isometric if for every bounded functional ϕ on the CQG, $(\mathrm{id} \otimes \phi) \circ \alpha$ maps $C^\infty(M)$ into itself and commutes with $\mathcal{L}$ there. We have:

Theorem

There exists a universal object (denoted by $QISO^{\mathcal{L}}(M)$) in the category of CQG acting isometrically on M.

No quantum isometry for compact connected manifolds

Now we focus on CQG actions on classical manifolds.

- Explicit computations for spheres, tori, G/K for compact connected semisimple G quotiented by suitable subgroups, showed that $QISO^{\mathcal{L}}(M) = C(ISo(M))$, i.e. there are no genuine quantum isometries.
- Banica, de Commer and Bhowmick showed that many known genuine CQG's including $SU_q(n)$ etc. cannot act faithfully isometrically on a connected compact manifold.
- This supported the no-go conjecture.
- There have been several other results, both in the algebraic and analytic set-up, which point towards the truth of this conjecture. We mention some of them here.

- it is verified by Goswami-Joardar under the additional condition that the action is isometric for some Riemannian metric on the manifold.
- Etingof and Walton obtained a somewhat similar result in the purely algebraic set up by proving that there cannot be any finite dimensional Hopf algebra having inner faithful action on a commutative domain.
- We should also mention the proof by A. L. Chirvasitu of non-existence of genuine quantum isometry in the metric space set-up for geodesic metric of negatively curved, compact connected Riemannian manifolds.

- If we could prove that any smooth action of a CQG can be made isometric for some Riemannian metric, the conjecture would be verified in full generality.
- we had announced in an archived article a proof of this fact (hence the truth of the conjecture in full generality) but there was a gap in the averaging argument.
- **Very recently, the speaker has come up with a solution to the above problem with averaging (thereby proving the general version of the conjecture) using techniques from classical probability theory, which will be briely sketched in this lecture.**

Smooth CQG action

- An action α of a CQG $\mathcal{Q}$ on M is called smooth if it maps $C^\infty(M)$ to $C^\infty(M, \mathcal{Q})$ and the span of $\alpha(C^\infty(M))(1 \otimes \mathcal{Q})$ is dense in $C^\infty(M, \mathcal{Q})$ in the natural Frechet topology.
- Just as in case of a C^*-action, given a smooth action α, following Podles, Soltan, Baum-de-Commer-Hajac, we can find a Frechet-dense unital $*$-subalgebra $\mathcal{C}_0$ of $C^\infty(M)$ on which α is an algebraic (co)-action of the dense Hopf $*$-algebra $\mathcal{Q}_0$ of $\mathcal{Q}$ mentioned above.
- We use Sweedler type notation $\alpha(f) = f_{(0)} \otimes f_{(1)}$ for $f \in \mathcal{C}_0$.
- We say that α preserves a Riemannian inner product $< \cdot, \cdot >$ on M if $< df_{(0)}, dg_{(0)} > \otimes f^*_{(1)} g_{(1)} = \alpha(< df, dg >)$ for all $f, g \in \mathcal{C}_0$.
- Isometric actions, i.e. actions commuting with Laplacian, do preserve the corresponding Riemannain inner product.

Theorem

For a smooth action α, the following are equivalent:
(i) $\forall x \in M$, the algebra $\mathcal{Q}_x$ generated by $(\chi \otimes \mathrm{id})\alpha(f)$ and $\alpha(g)$, where χ is any smooth vector field and $f, g \in C^\infty(M)$, is commutative.
(ii) α preserves some Riemannian inner product on the manifold.

Theorem

Let $\mathcal{Q}$ be a CQG which has a faithful, isometric action on a compact, connected, smooth Riemannian manifold. Then $\mathcal{Q}$ must be classical, i.e. $C(G)$ for some compact group G.
In other words, the quantum isometry group of any compact connected Riemannian manifold M is $C(Iso(M))$.

This is proved in Goswami-Joardar (GAFA).

Averaging a smooth action

Let us fix a compact smooth connected manifold M, a faithful, smooth action α of a CQG $\mathcal{Q}$ on it, with a Frechet dense (unital, $*$)-subalgebra $\mathcal{C}_0$ of $C^\infty(M)$ on which α is algebraic. Let us also fix some Riemannian metric $<< \cdot, \cdot >>$ on M and let $\mathcal{L}$ be the corresponding (Hodge) Laplacian. We claim that we can 'average' this Laplacian to obtain a new operator which corresponds to a new Riemannian metric which is preserved by the action. We state a classical fact.

Proposition

Let $\mathcal{L} : C^\infty(M) \to C(M)$ be a linear map satisfying $\mathcal{L}(1) = 0$, $\mathcal{L}(\bar{f}) = \overline{\mathcal{L}(f)}$ and (i) locality, i.e. for any $x \in M$ and any $f \in C^\infty(M)$ such that $f(y) = 0$ for all y in an open neighbourhood of x, we must have $\mathcal{L}(f)(x) = 0$;
(ii) conditionally positive definiteness, i.e. $\forall f_1, \ldots, f_k \in C^\infty(M)$, $k \geq 1$ and $x \in M$, $((k_{\mathcal{L}}(f_i, f_j)(x)))$ is a nonnegative definite, where $k_{\mathcal{L}}(f, g) := \mathcal{L}(\bar{f}g) - \mathcal{L}(\bar{f})g - \bar{f}\mathcal{L}(g)$;
(iii) non-degeneracy, i.e. , $\forall x \in M$ there is a choice of local coordinates $f_1, \ldots, f_m$ around x for which $k_{\mathcal{L}}(f_i, f_j) \in C^\infty(M)$ for $i, j = 1, \ldots, m$ and $((k_{\mathcal{L}}(f_i, f_j)(x)))$ is invertible.
Then there is a unique Riemannian structure $< \cdot, \cdot >$ on M such that $< df, dg >_x = k_{\mathcal{L}}(f, g)(x)$ for all real valued C^∞ functions f, g.

Some generalities about a smooth action:

- Any CQG $\mathcal{Q}$ acting faithfully on $C(M)$ must be of Kac type, i.e the Haar state h is tracial and the antipode κ is norm-bounded on the reduced CQG $\mathcal{Q}_r$. So, from now on, w.l.g we assume $\mathcal{Q} = \mathcal{Q}_r$ and $\alpha = \alpha_r$.
- The nuclearity of $C^\infty(M)$ gives a continuous extension of the map $m : \mathcal{Q}_x^\infty \otimes_{\rm alg} \mathcal{Q} \to \mathcal{Q}$ (where $\mathcal{Q}_x^\infty$ is the nuclear space $\alpha_x(C^\infty(M))$, $\alpha_x := (\mathrm{ev}_x \otimes \mathrm{id}) \circ \alpha)$ to their (unique) topological tensor product. Using this we get a continuous map $\beta := m \circ (\mathrm{id} \otimes \kappa) \circ \Delta : \mathcal{Q}_x^\infty \to \mathcal{Q}$ which extends $\epsilon(\cdot)1$ on $\alpha_x(\mathcal{C}_0)$. This proves injectivity of α on $C^\infty(M)$ and then by some more standard arguments, injectivity of α follows.
- This implies the existence of a faithful α-invariant Borel probability measure on M, hence $\alpha = \mathrm{ad}_U$ for some unitary representation U on $L^2(M, \mu)$. Fix one such U.

Denote by M_F and M_f the operators of left multiplication by F (respectively f) on the Hilbert $\mathcal{Q}$-module $L^2(M,\mu)\otimes\mathcal{Q}$ (respectively $L^2(M,\mu)$). Most often we may write simply F or f for M_F or M_f respectively by making slight abuse of notation.

Lemma

For $F \in \mathcal{C}_0 \otimes_{\rm alg} \mathcal{Q}_0 \subset C^\infty(M,\mathcal{Q})$, we have

$$({\rm id}\otimes h)(U^{-1}M_F U) = M_{F^\sharp},$$

where $F^\sharp = ({\rm id}\otimes h)(U^{-1}(F)) \in \mathcal{C}_0$.

Using this, we can prove that the map $F \mapsto \Psi(F) := ({\rm id}\otimes h)(U^{-1}M_F U)$ extends to a unital completely positive map from $C(M,\mathcal{Q})$ to $C(M)$ satisfying $\alpha\circ\Psi = (\Psi\otimes{\rm id})\circ({\rm id}\otimes\Delta)$ (invariance). In particular, ${\rm ev}_p\circ({\rm id}\otimes h)(U^{-1}\cdot U)$ extends to a well-defined state on $C(M)$.

Define

$$\hat{\mathcal{L}}(f) = (\mathrm{id} \otimes h)\left(U^{-1}((\mathcal{L} \otimes \mathrm{id})(\alpha(f))U\right)$$

for $f \in C^\infty(M)$. Our goal is to prove that $\mathcal{L}$ is a nondegenerate, conditionally completely positive, local operator and hence by the proposition mentioned before, induces a Riemannian structure. This Riemannian structure is easily seen to be α invariant from the definition of $\mathcal{L}$ and the invariance of the map Ψ. It is also not difficult to prove the conditionally complete positivity and nondegeneracy. The main challenge is to prove the locality, as α is not a local map. We do this by adapting a probabilistic proof using stopping time. To this, end, we recall some basics of Brownian motion on manifolds.

Brownian motion and other probabilistic basics

Let $M \subset \mathbb{R}^n$ isometrically and let X_t to be the unique solution of the stochastic differential equation $dX_t = \sum_{i=1}^n X_t P_i(X_t) \circ dW_i(t), \quad X_0 \in M$, where $P_i(x)$ denotes the projection of the i-th coordinate unit vector of $\mathbb{R}^n$ on the tangent space $T_x M$ and $(W_1(t), \ldots, W_n(t))$ denotes the standard Brownian motion of $\mathbb{R}^n$, defined on a probability space $(\Omega, \mathcal{F}, P)$ (say) starting at the origin. For $x \in M$ let $X_t(x, \omega)$ be the the solution with $X_0 = x$. The heat semigroup generated by the Laplacian $\mathcal{L}$ is given by $T_t(f)(x) := \mathbb{E}_P(f(X_t(x, \cdot)))$. Moreover, for almost all ω in the sample space, the following hold:

(i) The random map $\gamma_t(\omega)$ given by $x \mapsto X_t(x, \omega)$ is a diffeomorphism for every t,

(ii) $(x, t) \mapsto X_t(x, \omega)$ is continuous.

(iii) $X_{t+s}(x, \omega) = X_t(X_s(x, \omega), \omega)$.

- A stopping time adapted to the Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$, where $\mathcal{F}_t = \sigma(X_s,\ 0 \leq s \leq t)$, is a $[0,\infty]$-valued random variable such that $\{\omega \ : \ \tau(\omega) \leq t\} \in \mathcal{F}_t$ for all $t \in \mathbb{R}_+$.
- A family $(M_t)_{t\geq 0}$ of a seprable Banach space Z-valued $(\mathcal{F}_t)$-adapted random variables is called a $(\mathcal{F}_t)$-martingale if $\mathbb{E}(\|M_t\|) < \infty$ for each t and $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$ (almost surely) for all $0 \leq s \leq t < \infty$.
- **Optional Sampling Theorem**: For a Z-valued martingale s.t. $t \mapsto M_t(\omega)$ is right continuous for almost all ω and any bounded stopping time τ, the process $M_{\tau\wedge t}$ is a martingale, where $a \wedge b := \min(a, b)$.

- Let $\mathcal{E} = \{U^{-1} M_F U : \ F \in C(M, \mathcal{Q})\} \equiv U^{-1} C(M, \mathcal{Q}) U$, clearly a separable C^*-algebra.
- For $F \in C(M, \mathcal{Q})$ let $j_t(F) \in L^\infty(\Omega, C(M, \mathcal{Q}))$ given by $j_t(F)(\omega)(x) = F(X_t(x, \omega))$.
- Define $J_t : C(M, \mathcal{Q}) \to L^\infty(\Omega, \mathcal{E})$ by $J_t(F)(\omega) = U^{-1} j_t(F)(\omega) U$, vieweing $j_t(F)(\omega)$ as left multiplication operator.
- Let $\tilde{T}_t := T_t \otimes \mathrm{id}_{\mathcal{Q}}$, $\tilde{\mathcal{L}} = \mathcal{L} \otimes \mathrm{id}$ on $C^\infty(M, \mathcal{Q})$.
- Define a unital $*$-homomorphism $\Pi_t : C(M) \to L^\infty(\Omega, \mathcal{E})$ by $\Pi_t(f) = J_t(\alpha(f))$. Using $\mathbb{E}_s \circ j_{s+t} = j_s \circ \tilde{T}_t$ for $s, t \geq 0$, we prove the the following is a right continuous $\mathcal{F}_t$-martingale:

$$M_t^f = \Pi_t(f) - \int_0^t J_s(\tilde{\mathcal{L}}(\alpha(f)))ds.$$

Let $Y_i(t) = \Pi_t(x_i)$. Observe $Y_i(0) = y_i \otimes 1$.
To show the locality of $\hat{\mathcal{L}}$ at $p = (p_1, \ldots, p_n)$ of $M \subset \mathbb{R}^n$, consider $f = \phi(x_1, \ldots, x_n)$, where ϕ is a smooth function on $\mathbb{R}^n$, such that ϕ vanishes on $E_\epsilon(p) \bigcap M$, where $E_\epsilon(p)$ is a cube of side-length $\epsilon_0 > 0$ around p. By the continuity of the Brownian flow, $t \mapsto J_t(F)(\omega)$ is norm continuous almost surely for any fixed $F \in C^\infty(M, \mathcal{Q})$. Choose $F = \tilde{\mathcal{L}}(\alpha(f))$ and define two stopping times $\tau''_\epsilon, \tau'_\epsilon$ $(\epsilon > 0)$:

$$\tau''_\epsilon := \inf\{t \geq 0 : \ \|J_t(F)(\omega) - J_0(F)(\omega)\| > \epsilon\},$$

$$\tau'_\epsilon := \inf\{t \geq 0 : \ \|Y_i(t,\omega) - y_i \otimes 1\| > \epsilon \text{ for some} i\}.$$

Let $\tau_\epsilon = \min(\tau'_\epsilon, \tau''_\epsilon, 1)$, which is a bounded stopping time. By Optional Sampling Theorem, $\mathbb{E}(M^f_{\tau_\epsilon}) = M^f_0 = f \otimes 1$.
By definition of τ_ϵ and continuity of the Brownian flow, $\|J_s(F)(\omega) - J_0(F)(\omega)\| \leq \epsilon$ for all $s \leq \tau_\epsilon$.

Hence we get

$$\lim_{\epsilon \to 0+} \frac{\mathbb{E}(\Pi_{\tau_\epsilon}(f)) - f \otimes 1}{\mathbb{E}(\tau_\epsilon)} = \lim_{\epsilon \to 0+} \frac{\mathbb{E}(\int_0^{\tau_\epsilon} J_s(F)ds)}{\mathbb{E}(\tau_\epsilon)} = U^{-1}\tilde{\mathcal{L}}(\alpha(f))U.$$

For a fixed t and ω, let us denote by $\mathcal{B}_{t,\omega}$ the commutative unital C^* algebra generated by

$$\{\Pi_t(f)(\omega), g \otimes 1, f, g \in C^\infty(M)\},$$

and let $\mathcal{S}$ be the set of states ζ on $\mathcal{B}_{t,\omega}$ which extends ev_p on $C(M) \otimes 1 \cong C(M) \subset \mathcal{B}_{t,\omega}$. It follows by standard arguments that any extreme point of $\mathcal{S}$ is a pure state of $\mathcal{B}_{t,\omega}$. i.e. a $*$-homomorphism. Now, recall that $(\mathrm{id} \otimes h)(\mathcal{B}_{t,\omega}) \subseteq C(M)$, so $\eta := (\mathrm{ev}_p \otimes h) \in \mathcal{S}$. Moreover,

$$\hat{\mathcal{L}}(f)(p) = \lim_{\epsilon \to 0+} \frac{\mathbb{E}(\eta(\Pi_{\tau_\epsilon}(f)))}{\mathbb{E}(\tau_\epsilon)},$$

as $f(p) = 0$.

We claim that $\eta(\Pi_{\tau_\epsilon}(f)) = 0$ for all sufficiently small ϵ. As $\eta \in \mathcal{S}$, it is enough to prove it for all extreme points of $\mathcal{S}$. Let ζ be any such extreme point, it follows from the continuity of the Brownian path that $|\zeta(Y_{\tau_\epsilon}(t) - y_i \otimes 1)| \leq \epsilon$ for all i. But $\zeta(y_i \otimes 1) = p_i$ by defintion of $\mathcal{S}$, hence the tuple $(\zeta(Y_1(\tau_\epsilon)), \ldots, \zeta(Y_n(\tau_\epsilon))) \in \mathbb{R}^n \in E_p(\epsilon)$ and as $\zeta \circ \Pi_{\tau_\epsilon}$ is a character of $C(M)$, there is some point $v = (v_1, \ldots, v_n) \in M \subset \mathbb{R}^n$ such that $\zeta \circ \Pi_{\tau_\epsilon}(f) = f(v)$ for all $f \in C(M)$. In particular, $(\zeta(Y_1(\tau_\epsilon)), \ldots, \zeta(Y_n(\tau_\epsilon))) = (v_1, \ldots, v_n) \in M$. Thus, $\zeta(\Pi_{\tau_\epsilon}(f)) = \phi(\zeta(Y_1(\tau_\epsilon)), \ldots, \zeta(Y_n(\tau_\epsilon))) = 0$ for all $\epsilon < \epsilon_0$, proving our claim that $\hat{\mathcal{L}}(f)(p) = 0$.

Conclusion : Any smooth action presrerves some Riemannian inner product on M, hence $\mathcal{Q}_x$ is commutative.

Proof of the conjecture

Now, the steps of the no-go theorem for QISO will go through almost verbatim. The following is crucial:

Lemma

Let W be a subset with nonempty interior of some Euclidean space $\mathbb{R}^N$. If a faithful action α of a CQG $\mathcal{Q}$ on X is affine in the sense that α leaves invariant the linear span of the coordinate functions and the constant function 1, then $\mathcal{Q} \cong C(G)$ for some group G.

- We can use the commutativity of $\mathcal{Q}_x$ to lift the action to a (smooth) action on the total space of the unit sphere of the cotangent bundle, and applying again the commutativity of the corresponding first order partial derivatives of this lifted action, we can conclude that the upto second order partial derivatives of the original action form a commutative algebra. We can go on like this.

- Commutativity of $\mathcal{Q}_x$ allows us to further lift α to a smooth action $\tilde{\alpha}$ (say) on $C^\infty(T(O_M))$, where $E = T(O_M)$ is the total space of the orthonormal frame bundle on M.
- Choose a Riemannian structure on E which is preserved by $\tilde{\alpha}$.
- As E is parallelizable, choose an embedding in some $\mathbb{R}^m$ with trivial normal bundle (w.r.t. the Riemannian metric chosen above) say $N(E)$, lift $\tilde{\alpha}$ as an inner product preserving action Φ on some suitable ϵ-neighbourhood W of E in the total space $N \equiv \mathbb{R}^m$ of $N(E)$.
- Using commutativity of partial derivates upto second order of Φ we prove that it is affine, hence prove that $\mathcal{Q}$ is commutative.

Step 4: Affine-ness of Φ

- Let $D_i^k = \frac{\partial}{\partial y_i}\Phi(y_k)$, $D_{ij}^k = \frac{\partial^2}{\partial y_i \partial y_j}\Phi(y_k)$, where $y_1, \ldots, y_m$ are the standard coordinates of $\mathbb{R}^m$. As $Int(W)$ is open connected, it suffices to show that $D_{ij}^k = 0$ for all k, i, j.
- As isometric actions satisfy second order commutativity, D_{ij}^k and D_m^l commute.
- By the isometry condition of Φ:

$$\sum_{l=1}^{N} D_i^l D_j^l = \delta_{ij} 1. \tag{1}$$

- Applying $\frac{\partial}{\partial y_k}$ to equation (1), and using the commutativity of D^l_{jk} and D^l_i's

$$\sum_{l=1}^{N} (D^l_{ik} D^l_j + D^l_{jk} D^l_i) = 0. \tag{2}$$

- $A_{n^2 \times n} \equiv ((A_{(ij),k}))$, with $A_{(ij),k} = D^k_{ij}$, $B_{n \times n} = ((B_{ij} = D^i_j))$, $C := AB$.
- From (2)

$$C_{(ik)j} + C_{(jk)i} = 0. \tag{3}$$

- As $C_{(ij)k} = C_{(ji)k}$ for all i, j, k, equation (3) gives

$$C_{(ik)j} = C_{(ki)j} = -C_{(ji)k} = -C_{(ij)k} = C_{(kj)i} = C_{(jk)i}.$$

- So again by equation (3), $C_{(ik)j} = 0$ for all i, j, k i.e. $C = 0$, hence $A = 0$ as B is unitary.