Homomorphisms of noncommutative Hardy algebras

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Bangalore, India, December 2018

Introduction

Given two operator algebras $\mathcal{A}$ and $\mathcal{B}$, every completely contractive homomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ induces a map Z from the (c.c) representations of $\mathcal{B}$ to the (c.c.) representations of $\mathcal{A}$ by composition.

For the class of Hardy algebras associated with correspondences we can describe the set of (c.c) representations.

We use this to study the some of the properties of Z when $\mathcal{A}$ and $\mathcal{B}$ are such Hardy algebras.

Preliminaries

For the basic constructs we need the following setup:

- M - a W^*-algebra.
- E - a W^*-correspondence over M. This means that E is a **bimodule** over M which is endowed with an M-**valued inner product** (making it a right-Hilbert C^*-module that is self dual). The left action of M on E is given by a unital, normal, *-homomorphism φ of M into the (W^*-) algebra of all bounded adjointable operators $\mathcal{L}(E)$ on E.

Examples

- (Basic Example) $M = \mathbb{C}$, $E = \mathbb{C}^d$, $d \geq 1$.
- $G = (G^0, G^1, r, s)$- a finite directed graph. $M = \ell^\infty(G^0)$, $E = \ell^\infty(G^1)$, $a\xi b(e) = a(r(e))\xi(e)b(s(e))$, $a, b \in M$,$\xi \in E$ $\langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \overline{\xi(e)}\eta(e)$, $\xi, \eta \in E$.
- M- arbitrary , $\alpha : M \to M$ a normal unital, endomorphism. $E = M$ with right action by multiplication, left action by $\varphi = \alpha$ and inner product $\langle \xi, \eta \rangle := \xi^* \eta$. Denote it ${}_\alpha M$.
- Φ is a normal, contractive, CP map on M. $E = M \otimes_\Phi M$ is the completion of $M \otimes M$ with $\langle a \otimes b, c \otimes d \rangle = b^* \Phi(a^* c) d$ and $c(a \otimes b)d = ca \otimes bd$.

Note: If σ is a representation of M on H, $E \otimes_\sigma H$ is a Hilbert space with $\langle \xi_1 \otimes h_1, \xi_2 \otimes h_2 \rangle = \langle h_1, \sigma(\langle \xi_1, \xi_2 \rangle_E) h_2 \rangle_H$.

Similarly, given two correspondences E and F over M, we can form the (internal) tensor product $E \otimes F$ by setting

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle = \langle f_1, \varphi(\langle e_1, e_2 \rangle_E) f_2 \rangle_F$$

$$\varphi_{E \otimes F}(a)(e \otimes f)b = \varphi_E(a)e \otimes fb$$

and applying an appropriate completion.
In particular we get "tensor powers" $E^{\otimes k}$.

Also, given a sequence $\{E_k\}$ of correspondences over M, the direct sum $E_1 \oplus E_2 \oplus E_3 \oplus \cdots$ is also a correspondence (after an appropriate completion).

For a correspondence E over M we define the Fock correspondence

$$\mathcal{F}(E) := M \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \cdots$$

For every $a \in M$ define the operator $\varphi_\infty(a)$ on $\mathcal{F}(E)$ by

$$\varphi_\infty(a)(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_n$$

and $\varphi_\infty(a)b = ab$.
For $\xi \in E$, define the "shift" (or "creation") operator T_ξ by

$$T_\xi(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n.$$

and $T_\xi b = \xi b$. So that T_ξ maps $E^{\otimes k}$ into $E^{\otimes(k+1)}$.

Definition

(1) The norm-closed algebra generated by $\varphi_\infty(M)$ and $\{T_\xi : \xi \in E\}$ will be called the **tensor algebra** of E and denoted $\mathcal{T}_+(E)$.

(2) The ultra-weak closure of $\mathcal{T}_+(E)$ will be called the **Hardy algebra** of E and denoted $H^\infty(E)$.

Examples

1. If $M = E = \mathbb{C}$, $\mathcal{F}(E) = \ell^2$, $\mathcal{T}_+(E) = A(\mathbb{D})$ and $H^\infty(E) = H^\infty(\mathbb{D})$.
2. If $M = \mathbb{C}$ and $E = \mathbb{C}^d$ then $\mathcal{F}(E) = \ell^2(\mathbb{F}_d^+)$, $\mathcal{T}_+(E)$ is Popescu's $\mathcal{A}_d$ and $H^\infty(E)$ is F_d^∞ (Popescu) or $\mathcal{L}_d$ (Davidson-Pitts). These algebras are generated by d shifts.

Representations

Theorem

Every completely contractive representation of $\mathcal{T}_+(E)$ on H is given by a pair $(\sigma, \mathfrak{z})$ where

1. *σ is a normal representation of M on $H = H_\sigma$. $(\sigma \in NRep(M))$*
2. *$\mathfrak{z} : E \otimes_\sigma H \to H$ is a contraction that satisfies*

$$\mathfrak{z}(\varphi(\cdot) \otimes I_H) = \sigma(\cdot)\mathfrak{z}.$$

We write $\sigma \times \mathfrak{z}$ for the representation and we have $(\sigma \times \mathfrak{z})(\varphi_\infty(a)) = \sigma(a)$ and $(\sigma \times \mathfrak{z})(T_\xi)h = \mathfrak{z}(\xi \otimes h)$ for $a \in M$, $\xi \in E$ and $h \in H$.

Write $\mathcal{I}(E, \sigma)$ for the intertwining space and $\mathbb{D}(E, \sigma)$ for the open unit ball there. Thus the c.c. representations of the tensor algebra are parameterized by the family $\{\overline{\mathbb{D}(E, \sigma)}\}_{\sigma \in NRep(M)}$.

Examples

(1) $M = E = \mathbb{C}$. So $\mathcal{T}_+(E) = A(\mathbb{D})$, σ is the trivial representation on H, $E \otimes H = H$ and $\mathbb{D}(E,\sigma)$ is the (open) unit ball in $B(H_\sigma)$.

(2) $M = \mathbb{C}$, $E = \mathbb{C}^d$. $\mathcal{T}_+(E) = \mathcal{A}_d$ (Popescu's algebra) and $\mathbb{D}(E,\sigma)$ is the (open) unit ball in $B(\mathbb{C}^d \otimes H, H)$. Thus the c.c. representations are parameterized by row contractions $(T_1, \ldots, T_d)$.

(3) M general, $E =_\alpha M$ for an automorphism α.
$\mathcal{T}_+(E) =$ the analytic crossed product.
The intertwining space can be identified with $\{X \in B(H) : \ \sigma(\alpha(T))X = X\sigma(T), T \in B(H)\}$ and the c.c. representations are $\sigma \times \mathfrak{z}$ where $\mathfrak{z}$ is a contraction there.

Representations of $H^\infty(E)$

The representations of $H^\infty(E)$ are given by the representations of $\mathcal{T}_+(E)$ that extend to an ultraweakly continuous representations of $H^\infty(E)$.

For a given σ, we write $\mathcal{AC}(E,\sigma)$ for the set of all $\mathfrak{z} \in \overline{\mathbb{D}(E,\sigma)}$ such that $\sigma \times \mathfrak{z}$ is a representation of $H^\infty(E)$.

We have

Theorem

$$\mathbb{D}(E,\sigma) \subseteq \mathcal{AC}(E,\sigma) \subseteq \overline{\mathbb{D}(E,\sigma)}.$$

Example

When $M = E = \mathbb{C}$, $H^\infty(E) = H^\infty(\mathbb{D})$ and $\mathcal{AC}(E,\sigma)$ is the set of all contractions in $B(H_\sigma)$ that have an H^∞-functional calculus.

Fix σ. Given $\mathfrak{z} \in \mathcal{AC}(E, \sigma)$ and $X \in H^\infty(E)$, we write

$$\widehat{X}(\mathfrak{z}) = (\sigma \times \mathfrak{z})(X).$$

In this way we view elements of $H^\infty(E)$ as (operator valued) functions (on $\mathcal{AC}(E, \sigma)$).
(If σ is not fixed, we write $\widehat{X_\sigma}$ instead of $\widehat{X}$.)

NP for homomorphisms

From now on we consider the following **SETUP**.
M is a W^*-algebra and $M_0 \subseteq M$ is a sub W^*-algebra. Let E be a W^*-correspondence over M and F is a W^*-correspondence over M_0. Fix a normal representation σ of M and σ_0 is the restriction of σ to M_0.
Then
Every completely contractive ultraweakly continuous homomorphism $\alpha : H^\infty(F) \to H^\infty(E)$ that maps $\varphi_{F\infty}(a)$ to $\varphi_{E\infty}(a)$ $(a \in M_0)$ gives rise to a map from the c.c. uw-c representations of $H^\infty(E)$ to the c.c. uw-c representations of $H^\infty(F)$ (by composition). I. e.,

$$\sigma \times \mathfrak{z} \mapsto (\sigma \times \mathfrak{z}) \circ \alpha = \sigma_0 \times \mathfrak{w}$$

where $\mathfrak{z} \in \mathcal{AC}(E, \sigma)$ and $\mathfrak{w} \in \mathcal{AC}(F, \sigma_0)$. **What can be said about the map $Z : \mathfrak{z} \mapsto \mathfrak{w}$?**

Theorem

Suppose that n distinct points $\mathfrak{z}_1, \ldots, \mathfrak{z}_n$ are given in $\mathbb{D}(E, \sigma)$ and n points $\mathfrak{w}_1, \ldots, \mathfrak{w}_n$ are given in $\mathcal{I}(F, \sigma_0)$. Define the map Φ_η on $M_n(\sigma(M)')$ by the formula $\Phi_{\mathfrak{z}}((a_{ij})) = (\mathfrak{z}_i(I_E \otimes a_{ij})\mathfrak{z}_j^*)$ and, similarly, $\Phi_{\mathfrak{w}}((a_{ij})) = (\mathfrak{w}_i(I_F \otimes a_{ij})\mathfrak{w}_j^*)$ (on $M_n(\sigma_0(M_0))'$).
Then the Pick operator $P := (I - \Phi_{\mathfrak{w}}) \circ (I - \Phi_{\mathfrak{z}})^{-1}$ is completely positive if and only if there is a completely contractive homomorphism $\alpha : \mathcal{T}_+(F) \to H^\infty(E)$ such that $\alpha(\varphi_{F\infty}(a)) = \varphi_{E\infty}(a)$ for $a \in M_0$ and, for every $X \in \mathcal{T}_+(F)$,

$$\widehat{\alpha(X)}(\mathfrak{z}_j) = \widehat{X}(\mathfrak{w}_j)$$

(Equivalently, Z maps $\mathfrak{z}_j$ to $\mathfrak{w}_j$).
for every $1 \leq j \leq n$. Moreover, if $\mathfrak{w}_j \in \mathcal{AC}(F, \sigma_0)$ and $(I - r\Phi_{\mathfrak{w}}) \circ (I - \Phi_{\mathfrak{z}})^{-1}$ is completely positive for some $r > 1$, then α extends to a completely contractive ultraweakly homomorphism on $H^\infty(F)$.

Theorem

Let H be a Hilbert space and fix an operator $b \in B(H)$ that has an H^∞-calculus (e.g. $||b|| < 1$) and a row contraction $A = (a_1, \ldots, a_d)$ where all a_i are operators in $B(H)$. Then the following statements are equivalent.

1. *(1) There are functions $f_1, \ldots, f_d$ in $H^\infty(\mathbb{D})$ such that $\sum_k |f_k|^2 \leq 1$ and, for every k, $f_k(b) = a_k$.*
2. *(2) Whenever $p \geq 1$ and $x = (x_{ij})$, in $M_p(B(H))$, satisfies $x \geq 0$, $(bx_{ij}b^*) \leq x$ and $(b^n x_{ij} b^{*n}) \searrow 0$, we have $\sum_k (a_k x_{ij} a_k^*) \leq x$.*
   If $||b|| < 1$, *the condition $(b^n x_{ij} b^{*n}) \to 0$ (in (2)) is automatic and (1) and (2) are also equivalent to*
3. *(3) The map $(id - AdA) \circ (id - Adb)^{-1} : B(H) \to B(H)$ is completely positive.*

In particular, condition (2) (or (3)) implies that all the a_k's are contained in the ultraweakly closed algebra generated by b (since this, clearly, follows from (1)).

The proof: Let $M = M_0 = \mathbb{C}$, $F = \mathbb{C}^d$, $E = \mathbb{C}$ (for $d < \infty$) and σ is the representation of $\mathbb{C}$ on a Hilbert space H. It follows that $\mathcal{I}(E, \sigma) = B(H)$ and $\mathcal{I}(F, \sigma_0) = B(\mathbb{C}^d \otimes H, H)$.
We apply the theorem to this case.
Since $\mathcal{T}_+(F)$ is the algebra $\mathcal{T}_+(\mathbb{C}^d)$, generated by the d shifts, $S_1, \ldots, S_d$ on the Fock space of $\mathbb{C}^d$, and $H^\infty(E) = H^\infty(\mathbb{D})$, the existence of a unital completely contractive homomorphism α from $\mathcal{T}_+(F)$ to $H^\infty(E)$ is equivalent to the existence of a row contraction of elements in $H^\infty(\mathbb{D})$, $(f_1, \ldots, f_d)$ (where $\alpha(S_i) = f_i$).
The condition $\widehat{\alpha(X)}(\mathfrak{z}) = \widehat{X}(\mathfrak{w})$ in the theorem (note that $m = 1$ here), when applied to $X = S_i$, $\mathfrak{z} = b$ and $\mathfrak{w} = A$, is $f_i(b) = a_i$.
Note also that $\Phi_\eta = Adb$ (i.e. $\Phi_\eta(x) = bxb^*$) and $\Phi_\xi(x) = AdA(x) = \sum_{i=1}^d a_i x a_i^*$.
Thus, applying the theorem , completes the proof.

Theorem

Let H_0 and H_1 be two Hilbert spaces and fix $b \in B(H_1)$ and $A = (a_1, \ldots, a_d)$ with $a_i \in B(H_0 \otimes H_1)$ and such that $||b|| < 1$ .Then the following statements are equivalent.

1. *(1) There are $f_1, \ldots, f_d$ in $H^\infty(\mathbb{D}, B(H_0))$ such that $\sum_k f_k f_k^* \leq I$ and such that, for every k, $\widehat{f_k}(b) = a_k$.*
2. *(2) For every $p \geq 1$ and $x = (x_{ij}) \geq 0$ in $M_p(B(H_1))$ that satisfies $(b x_{ij} b^*) \leq x$, we have $\sum_k (a_k(I_{H_0} \otimes x_{ij}) a_k^*) \leq I_{H_0} \otimes x$.*

Schur class maps

For $\mathfrak{z}, \mathfrak{w} \in \mathbb{D}(E, \sigma)$ we write $\theta_{\mathfrak{z},\mathfrak{w}} : \sigma(M)' \to \sigma(M)'$ for the map $\theta_{\mathfrak{z},\mathfrak{w}}(a) = \mathfrak{z}(I_E \otimes a)\mathfrak{w}^*$.

Definition

A map $Z : \Omega \subseteq \mathbb{D}(E, \sigma) \to \mathcal{I}(F, \sigma_0)$ is said to be a **Schur class map** if the kernel

$$k_Z(\mathfrak{z}, \mathfrak{w}) = (id - \theta_{Z(\mathfrak{z}),Z(\mathfrak{w})}) \circ (id - \theta_{\mathfrak{z},\mathfrak{z}})^{-1}$$

is a CP kernel; i.e., for every $\mathfrak{z}_1, \ldots, \mathfrak{z}_k \in \Omega$, the map from $M_k(\sigma(M)')$ to $M_k(\sigma_0(M_0)')$ defined by the $k \times k$ matrix of maps

$$(id - \theta_{Z(\mathfrak{z}_i),Z(\mathfrak{z}_j)}) \circ (id - \theta_{\mathfrak{z}_i,\mathfrak{z}_j})^{-1}$$

is completely positive.

Theorem

Let M, M_0, E, F, σ and σ_0 be as above with σ faithful. Suppose Ω is a subset of $\mathbb{D}(E,\sigma)$ and $Z : \Omega \to \mathcal{I}(F,\sigma_0)$ is a **Schur class map**. *Then there is a Hilbert space H and a normal representation τ of $\sigma(M)'$ on H and operators A, B, C and D satisfying appropriate intertwining properties and such that the operator matrix*

$$W = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} H \\ F \otimes_{\sigma_0} H_\sigma \end{pmatrix} \to \begin{pmatrix} \mathcal{I}(E,\sigma)^* \otimes_\tau H \\ H_\sigma \end{pmatrix}$$

is a co-isometry (and, if E is full, it is a unitary operator) and, for every $\mathfrak{z} \in \Omega$, Z satisfies the **realization formula**

$$Z(\mathfrak{z}) = D + C(I - \mathfrak{z}A)^{-1}\mathfrak{z}B.$$

Theorem

Suppose E is a full W^*-correspondence over the W^*-algebra M and F, M_0, σ and σ_0 be as above with σ faithful. Suppose $Z : \mathbb{D}(E,\sigma) \to \mathcal{I}(F,\sigma_0)$ has the **realization property** *(in particular, if it is a Schur class map). Then* **there is a completely contractive homomorphism** $\alpha : \mathcal{T}_+(F) \to H^\infty(E)$ *such that, for every $X \in \mathcal{T}_+(F)$ and every $\mathfrak{z} \in \mathbb{D}(E,\sigma)$,*

$$\widehat{\alpha(X)}(\mathfrak{z}) = \widehat{X}(Z(\mathfrak{z}))$$

Theorem

With $M, M_0, E, F, \sigma, \sigma_0$ as above, if $\alpha : \mathcal{T}_+(F) \to H^\infty(E)$ is a **completely contractive homomorphism** *whose restriction to $\varphi_{F^\infty}(M_0)$ maps $\varphi_{F^\infty}(a)$ to $\varphi_{E^\infty}(a)$ $(a \in M_0)$ then* **there is a Schur class function** $Z : \mathbb{D}(E^\sigma) \to \mathcal{I}(F, \sigma_0)$ *such that for every $X \in \mathcal{T}_+(F)$ and every $\mathfrak{z} \in \mathbb{D}(E, \sigma)$,*

$$\widehat{\alpha(X)}(\mathfrak{z}) = \widehat{X}(Z(\mathfrak{z})).$$

Corollary

Every Schur class map Z_0 defined on a finite subset of $\mathbb{D}(E, \sigma)$ can be extended to a Schur class map on $\mathbb{D}(E, \sigma)$.

Consider all representations: Matricial maps

Given $X \in H^\infty(E)$, we define a family $\{\widehat{X}_\sigma\}_{\sigma \in NRep(M)}$ of (operator valued) functions.
Each function $\widehat{X}_\sigma$ is defined on $\mathcal{AC}(E, \sigma)$ (or on $\mathbb{D}(E, \sigma)$) and takes values in $B(H_\sigma)$:

$$\widehat{X}_\sigma(\mathfrak{z}) = (\sigma \times \mathfrak{z})(X).$$

Here $NRep(M)$ is the set of all normal representations of M.
Note that the family of domains (either $\{\mathcal{AC}(\sigma)\}$ or $\{\mathbb{D}(E, \sigma)\}$) is a matricial family in the following sense.

Definition

A family of sets $\{\mathcal{U}(\sigma)\}_{\sigma \in NRep(M)}$, with $\mathcal{U}(\sigma) \subseteq \mathcal{I}(E, \sigma)$, satisfying $\mathcal{U}(\sigma) \oplus \mathcal{U}(\tau) \subseteq \mathcal{U}(\sigma \oplus \tau)$ is called a *matricial family* of sets.

Definition

Suppose $\{\mathcal{U}(\sigma)\}_{\sigma \in NRep(M)}$ is a matricial family of sets and suppose that for each $\sigma \in NRep(M)$, $f_\sigma : \mathcal{U}(\sigma) \to B(H_\sigma)$ is a function. We say that $f := \{f_\sigma\}_{\sigma \in NRep(M)}$ is a *matricial family of functions* in case

$$Cf_\sigma(\mathfrak{z}) = f_\tau(\mathfrak{w})C \tag{1}$$

for every $\mathfrak{z} \in \mathcal{U}(\sigma)$, every $\mathfrak{w} \in \mathcal{U}(\tau)$ and every $C \in \mathcal{I}(\sigma \times \mathfrak{z}, \tau \times \mathfrak{w})$ (equivalently, $C \in \mathcal{I}(\sigma, \tau)$ and $C\mathfrak{z} = \mathfrak{w}(I_E \otimes C)$).

Theorem

Suppose that $f = \{f_\sigma\}_\sigma$ is a family of maps, with $f_\sigma : \mathcal{AC}(E, \sigma) \to \mathcal{AC}(F, \sigma_0)$. Then f is a matricial family of maps (that is, preserves intertwiners) if and only if there is an ultraweakly continuous homomorphism $\alpha : H^\infty(F) \to H^\infty(E)$ such that for every $\mathfrak{z} \in \mathcal{AC}(E, \sigma)$ and every $X \in H^\infty(F)$,

$$\widehat{\alpha(X)}(\mathfrak{z}) = \widehat{X}(f_\sigma(\mathfrak{z})). \tag{2}$$

Thank You !