The Commutant mod a Normed Ideal of an n-tuple of Operators

Dan-Virgil Voiculescu
UC Berkeley

The obstruction to quasicentral approximate units, measured by the number $k_J(\tau)$ plays a key role in normed ideal perturbations. Recently $\mathcal{E}(\tau; J)$, the commutant mod the normed ideal is emerging as an object of interest.

$\mathcal{H}$ complex separable ∞ dim

$(\mathfrak{I}, |\ |_{\mathfrak{I}})$ normed ideal of compact operators

$(\mathcal{C}_p, |\ |_p)$ p-class, $|T|_p = \left(\sum_j s_j^p\right)^{1/p}$

$(\mathcal{C}_p^-, |\ |_p^-)$ Lorentz $(p,1)$

$$|T|_p^- = \sum_j s_j\, j^{-1+1/p},\quad (1 \leq p \leq \infty)$$

$s_1 \geq s_2 \geq \cdots$ eigenvalues of $(T^*T)^{1/2}$

$\tau = (T_1, \dots, T_n)$ n-tuple bdd. operators

$\mathcal{R}_1^+ = \{ A \mid 0 \leq A \leq I,\ A \text{ finite rank} \}$

$$k_{\mathfrak{J}}(\tau) = \liminf_{A \in \mathcal{R}_1^+} \max_{1 \leq j \leq n} |[A, T_j]|_{\mathfrak{J}}$$

$$k_{\mathfrak{J}}(\tau) = 0 \iff A_n \uparrow I,\ A_n \in \mathcal{R}_1^+,$$
$$|[A_n, T_j]|_{\mathfrak{J}} \to 0,\quad 1 \leq j \leq n$$

(quasicentral approximate unit for τ relative to $\mathfrak{J}$)

$\mathfrak{J} = \mathcal{C}_p \quad k_p(\tau), \quad \mathfrak{J} = \mathcal{C}_p^- \quad k_p^-(\tau)$

$k_J(\tau)$ "Size-J dimensional measure of τ"

p-dimensional $\sim$ $J = \mathcal{C}_p^-$

$k_p(\tau) \in \{0, \infty\}$ if $1 < p$ $\quad (\mathcal{C}_1 = \mathcal{C}_1^-)$

τ commuting n-tuple of Hermitian ops.

$$\left(k_n^-(\tau)\right)^n = \gamma_n \int_{\mathbb{R}^n} m(s)\, d\lambda(s)$$

multiplicity function of Lebesgue abs. cont. part of spectral measure

$k_n^-(\tau)=0 \iff$ spectral measure of τ singular wrt Lebesgue

In general $k_p^-(\tau)$ as function of p decreasing

$$0<k_{p_0}^-(\tau)<\infty \implies \begin{array}{ll} k_p^-(\tau)=\infty & p<p_0 \\ k_p^-(\tau)=0 & p>p_0 \end{array}$$

$$\tau-\tau'\in J \implies k_J(\tau)=k_J(\tau')$$

$p = \infty$ $\quad k_\infty^-(\tau)$

$k_\infty^-(\tau) \leq 2 \|\tau\| \log(2n-1)$

$k_\infty^-(\tau \otimes I_{\mathcal{H}_1}) = k_\infty^-(\tau)$

$J \supset \mathcal{C}_\infty^-, \; J \neq \mathcal{C}_\infty^- \Rightarrow k_J(\tau) = 0$
(all τ)

$S_1 \ldots S_n$ $\quad k_\infty^-(S_1, \ldots, S_n) = \log n$

creations by $e_1 \ldots e_n$
on $\mathcal{T}(\mathbb{C}^n)$
(extended Cuntz)

$\bar{k}_\infty$ and entropy

1°. T measure preserving ergodic automorphism of (Ω, Σ, μ), $\mu(X) = 1$

U_T induced unitary in L^2

Φ multiplications in L^2 by meas. functions taking finite # of values

$$\mathcal{H}_p(T) = \sup_{\substack{\varphi \subset \Phi \\ \text{finite}}} \bar{k}_\infty(\varphi \cup \{U_T\})$$

$$\mathcal{H}_p(T) \asymp h(T)$$

Kolmogorov-Sinai entropy

2° μ finitary probability measure on group G with finite generator $g_1, \dots, g_n$

$$h(G,\mu) > 0 \Longrightarrow k_\infty^-(\lambda(g_1), \cdots, \lambda(g_n)) > 0$$

Avez entropy of random walk

left regular rep.

Further results on k_∞^- for

Gromov hyperbolic groups

entropy of subshifts

in Rui Okayasu papers

G finitely generated group K generators

$$k_g(\lambda(K)) = \begin{cases} 0 \\ \text{finite} \\ \infty \end{cases}$$ does not depend on choice of K

(generalizes Yamasaki's p-hyper/para-bolicity)

$$k_\infty^-(\lambda(K)) = 0 \implies G \text{ supramenable}$$

(recent result uses Kellerhals-Monod-Rørdam)

Problem: G supramenable $\overset{?}{\Longrightarrow} h_\infty^-(\lambda(K))=0$

(i.e. G supramenable $\overset{?}{\Longleftrightarrow} h_\infty^-(\lambda(K))=0$)

Existence of positive entropy finitary random walk on supramenable G would imply negative answer.

Problem: G generator K, G' generator K'

$$h_\infty^-(\lambda(K)) = h_\infty^-(\lambda(K'))=0 \overset{?}{\Longrightarrow} h_\infty^-(\lambda(K)\otimes\lambda(K')) = 0$$

Uses of $k_J(\tau)$

Adaptation to normed ideals of the Noncommutative Weyl-v. Neumann Type Theorem

A C^*-alg. with $\mathbb{1}$, $X_1, \dots X_n$ generator

ρ_1, ρ_2 $*$-representations on $\mathcal{H}$, $\rho_j(A) \cap \mathcal{K} = \{0\}$, $j=1,2$

$k_J(\rho_j(\{X_1, \dots, X_n\})) = 0$, $j = 1,2$.

$$\Downarrow$$

$\exists U$ unitary $|U\rho_1(X_k)U^* - \rho_2(X_k)|_J < \varepsilon$

$k = 1, \dots, n$

<u>Cor.</u> N normal $\Rightarrow N = \underset{\text{diagonal}}{D} + \mathcal{C}_2$

$[A = C(\underset{\mathbb{R}^2}{\widehat{K}})$, X_1, X_2 coordinate functions$]$

Generalized singular and absolutely continuous subspaces of τ w.r.t. J

$\mathcal{H} = \mathcal{H}_s(\tau; J) \oplus \mathcal{H}_a(\tau; J)$ τ-reducing

$\mathcal{H}_s(\tau; J)$ largest τ-reducing subspace $\mathcal{K}$ so that $k_J(\tau|\mathcal{K}) = 0$.

τ n-tuple of commuting Hermitian ops
$\mathcal{J} = \mathcal{C}_n^-$ then:

$\mathcal{H}_s(\tau; \mathcal{C}_n^-) =$ Lebesgue singular subspace $\mathcal{H}_{sing}(\tau)$

$\mathcal{H}_a(\tau; \mathcal{C}_n^-) =$ Lebesgue absolutely cont. $\mathcal{H}_{ac}(\tau)$

$\tau - \tau' \in \mathcal{C}_n^-$ then

$\tau | \mathcal{H}_{ac}(\tau) \underset{\text{unitary}}{\sim} \tau' | \mathcal{H}_{ac}(\tau')$

for $n=1$ consequence of Kato-Rosenblum Thm
for general n proved using $k_n^-(\tau)$ machinery

The Banach algebras $\mathcal{E}(\tau; \mathcal{J})$

$\tau = \tau^* = (T_j)_{1 \le j \le n} \subset \mathcal{B}(\mathcal{H}),\ (\mathcal{J}, |\ |_{\mathcal{J}})$

$\mathcal{E}(\tau; \mathcal{J}) = \{X \in \mathcal{B}(\mathcal{H}) \mid [X, T_j] \in \mathcal{J},\ 1 \le j \le n\}$

$|||X||| = \|X\| + \max_{1 \le j \le n} |[X, T_j]|_{\mathcal{J}}$

Banach $*$-algebras with isometric involution

$\mathcal{K}(\tau; \mathcal{J}) = \mathcal{E}(\tau; \mathcal{J}) \cap \mathcal{K}$

closed 2-sided ideal in $\mathcal{E}(\tau; \mathcal{J})$

$\mathcal{E}/\mathcal{K}(\tau; \mathcal{J}) = \mathcal{E}(\tau; \mathcal{J}) / \mathcal{K}(\tau; \mathcal{J})$

If $\mathcal{J} = \mathcal{K}$, $\mathcal{E}/_{\mathcal{K}}(\tau;\mathcal{K}) =$ Paschke dual of $C^*(\rho(\tau))$
ρ homomorphism to $\mathcal{B}/\mathcal{K}$

$\mathcal{E}(\tau;\mathcal{J})$ or $\mathcal{E}/_{\mathcal{K}}(\tau;\mathcal{J})$ <u>are not</u> in general somekind of smooth subalgebras of $\mathcal{E}(\tau;\mathcal{K})$ or $\mathcal{E}/_{\mathcal{K}}(\tau;\mathcal{K})$

Much richer K-theory, which reflects perturbation theory facts

τ n-tuple of commuting Hermitian operators

$\sigma(\tau) = [0,1]^n$ to simplify

which implies $K_0(\mathcal{E}(\tau;\mathcal{K})) = 0$.

$m_{ac}(\tau)$ = multiplicity of Lebesgue absolutely continuous part of τ
a.e. defined measurable function $[0,1]^n \longrightarrow \{0,1,2,\dots\infty\}$

$$\mathcal{F}(\tau) = K_0((\tau|\mathcal{H}_{ac}(\tau))')$$

$$\sim \quad f : [0,1]^n \longrightarrow \mathbb{Z}, \quad f|(m_{ac}(\tau))^{-1}(\infty) = 0$$

$$|f(x)| \leq C\, m_{ac}(\tau)(x)$$

a.e. etc.
measurable

1°. $n=1$, $\mathcal{J}=\mathcal{C}_1$

$$K_0(\mathcal{E}(T;\mathcal{C}_1)) \simeq \mathcal{F}(T)$$

$$[P]_0 \rightsquigarrow \mathrm{mac}(P(T\otimes I_n)P)\,\chi_{\mathrm{mac}(\tau)^{-1}([0,\infty))}$$

2°. $n=1$, $\mathcal{J}\neq\mathcal{C}_1$ (means $\mathcal{J}\supsetneqq\mathcal{C}_1$)

$$K_0(\mathcal{E}(T;\mathcal{J}))=0$$

3°. $n\geq 3$, $\mathcal{J}=\mathcal{C}_n^-$, assume $\mathcal{H}_{ac}(\tau)=\mathcal{H}$

$$K_0(\mathcal{E}(\tau;\mathcal{C}_n^-))=\mathcal{F}(\tau)\oplus \underset{\text{unknown}}{\mathcal{X}}$$

4°: $n=2$, $\mathcal{J}=\mathcal{C}_2$

$$K_0(\mathcal{E}(\tau;\mathcal{C}_2)) \longrightarrow L^1_{real}([0,1]^2, d\lambda)$$

$$[P]_0 \rightsquigarrow g_{P(T_1+iT_2)P} \quad \text{Pincus principal function}$$

nontrivial homomorphism
infinite rank group in range

Homomorphisms in 1°, 3°, 4° "canonical":
do not depend on replacing
τ by τ', $\tau \equiv \tau' \bmod \mathcal{J}$.

$\mathcal{E}/\mathcal{K}(\tau;\mathcal{J})$

many similarities between

$\mathcal{K}$, $\mathcal{B}$, $\mathcal{B}/\mathcal{K}$ and

$\mathcal{K}(\tau;\mathcal{J})$, $\mathcal{E}(\tau;\mathcal{J})$, $\mathcal{E}/\mathcal{K}(\tau;\mathcal{J})$.

- Assume finite rank operators $\mathcal{R}$ dense in $\mathcal{J}$ and $k_{\mathcal{J}}(\tau)=0$ then $\mathcal{R}$ dense in $\mathcal{K}(\tau;\mathcal{J})$ and $\mathcal{E}/\mathcal{K}(\tau;\mathcal{J})$ isometrically isomorphic to C^*-subalgebra in $\mathcal{B}/\mathcal{K}$.

- Assume $\mathcal{R}$ dense in $\mathcal{J}$ and $k_{\mathcal{J}}(\tau) = 0$ then $\mathcal{E}/\mathcal{K}(\tau;\mathcal{J})$ is countably degree-1 saturated (in the sense of Farah-Hart)

- Assume $\mathcal{R}$ dense in $\mathcal{J}$ and $k_{\mathcal{J}}(\tau) < \infty$ then $\mathcal{R}$ dense in $\mathcal{K}(\tau;\mathcal{J})$ and $\mathcal{E}/\mathcal{K}(\tau;\mathcal{J})$ isomorphic as Banach-$*$algebra with C^*-subalgebra of $\mathcal{B}/\mathcal{K}$.

- Assume $\mathcal{R}$ dense in $\mathcal{J}$ and $\mathcal{J}^*$, and $k_{\mathcal{J}}(\tau) = 0$

  then $\mathcal{E}(\tau; \mathcal{J}) =$ bidual of $\mathcal{K}(\tau; \mathcal{J})$

  $\mathcal{E}(\tau; \mathcal{J}) = \mathcal{M}(\mathcal{K}(\tau; \mathcal{J}))$

- Assume $\mathcal{J}$ reflexive and $k_{\mathcal{J}}(\tau) = 0$

  then $\mathcal{E}(\tau; \mathcal{J})$ has unique predual.

Corollaries

- assume $\bar{\mathcal{R}} = \mathfrak{J}$, $k_{\mathfrak{J}}(\tau) = 0$, then bounded representations of countable amenable Γ into $\mathcal{E}/\mathcal{K}(\tau;\mathfrak{J})$ unitarizable
- assume $\bar{\mathcal{R}} = \mathfrak{J}$, $k_{\mathfrak{J}}(\tau) < \infty$ then $K_0(\mathcal{K}(\tau;\mathfrak{J})) \simeq \mathbb{Z}$, $K_1(\mathcal{K}(\tau;\mathfrak{J})) = 0$
- for every n-tuple τ $\mathcal{R}$ dense in $\mathcal{K}(\tau;\mathcal{C}_\infty^-)$ and $\mathcal{E}/\mathcal{K}(\tau;\mathcal{C}_\infty^-)$ isomorphic as Banach $*$-algebra to C^*-subalgebra in $\mathcal{B}/\mathcal{K}$

(Ref-1

1°. Almost normal operators mod Hilbert-Schmidt and the K-theory of the algebras $E\Lambda(\Omega)$ J. Noncommutative Geom. 8 (2014) no. 4, 1123–1145

2°. (with J. Bourgain) The essential centre of the mod a diagonalization ideal commutant of a n-tuple of commuting hermitian operators arXiv: 1309.1686

3°. Countable degree-1 saturation of certain C^*-algebras which are coronas of Banach algebras. Groups Geom. Dyn. 8 (2014) 985–1006

Ref-2

4°. Some C^*-algebras which are coronas of non-C^*-Banach algebras, J. of Geometry and Physics 105 (2016), 123–129

5°. A remark about supramenability and the Macaev norm, arXiv: 1605.02135

6°. K-theory and perturbations of absolutely continuous spectra arXiv: 1606.00520

Ref - 3

7°. Lebesgue Decomposition of functionals and unique preduals for commutants modulo normed ideals arXiv: 1608.07228

Surveys of old work

8° Perturbations of operators, connections with singular integrals, hyperbolicity and entropy. in Harmonic Analysis and Discrete Potential Theory (Frascati, 1991), 181-191 Plenum, New York, 1992

(Ref-4

9°. Hilbert space operators modulo normed ideals. Proceedings ICM Vol. 1,2 (Warsaw, 1983), 1041-1047 PWN, Warsaw 1984