# ON A THEOREM OF ABATZOGLOU FOR OPERATORS ON ABSTRACT $L$-SPACES

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ABSTRACT. Let $X$ be an abstract $L$-space and let $Y$ be any Banach spaces. Motivated by a classical result of T. J. Abatzoglou that describes smooth points of the space of operators on a Hilbert space, we give a characterization of very smooth points in the space of operators from $X$ to $Y$.

## 1. INTRODUCTION

Let $X, Y$ be a real Banach space. Let $\mathcal{L}(X, Y)$ denote the space of bounded linear operators. A non-zero vector $x_0 \in X$ is said to be a smooth point, if there is a unique unit vector $x^* \in X^*$ such that $x^*(x_0) = \|x_0\|$. It is said to be a very smooth point, if under the canonical embedding of $X$ in its bidual $X^{**}$, $x_0$ is a smooth point of $X^{**}$, see [6]. Let $S_X$ denote the set of unit vectors of $X$. For a Hilbert space $H$, by a classical result of Abatzoglou ([1]), $T \in \mathcal{L}(H)$ is a smooth point if and only if there exists a $x_0 \in S_H$ such that $T$ attains its norm only at $\pm x_0$ and $sup\{\|T(y)\| :< y, x_0 >= 0,\ y \in S_H\} < \|T\|$.

In this paper we are interested in formulating an abstract analogue of the above theorem for $\mathcal{L}(X, Y)$ when $X$ is an

2000 *Mathematics Subject Classification.* Primary 46 B 42, 47 L 05, 46 B20 .
*Key words and phrases.* Abstract $L$-spaces, Very smooth points, spaces of operators.

abstract $L$-space (see [4] Chapter 1 and Section 15 of Chapter 5). We show that if $T \in \mathcal{L}(X, Y)$ is a smooth point that attains its norm only at a $\pm x_0 \in S_X$, then $T(x_0)$ is a smooth point of $Y$ and if $P$ denotes the band projection associated with $span\{x_0\}$, then $\|T \circ (I - P)\| < \|T\|$. Conversely if $T \in \mathcal{L}(X, Y)$ attains its norm only at a $\pm x_0 \in S_X$, $T(x_0)$ is a smooth point of $Y$ and $\|T \circ (I - P)\| < \|T\|$ then $T$ is a smooth point of $\mathcal{L}(X, Y)$.

Under some additional assumptions of approximation property on $X^*$, we show that if $T \in \mathcal{L}(X, Y)$ is a very smooth point, then $T$ attains its norm only at a $\pm x_0 \in S_X$, and $T(x_0)$ is a very smooth point of $Y$. We recall that $x \in X$ is said to be Birkhoff-James orthogonal to $y \in X$ if $\|x\| \leq \|x + \lambda y\|$ for all real numbers $\lambda$. Since for any band projection $P$ in an $L$-space $X$, for $x \in \ker(P)$ and $y \in P(X)$, $\|x\| \leq \|x\| + |\lambda|\|y\| = \|x + \lambda y\|$, we also get the orthogonality aspect of Abatzoglou's result, now in terms of Birkhoff-James orthogonality.

This work was partially funded by the project 'Advances in noncommutative mathematics' of the Indian Statistical Institute.

## 2. Main Results

For a Banach space $X$, let $X_1$ denote the unit ball and $\partial_e X_1$ denote the set of extreme points. For $x^{**} \in X^{**}$ and $y^* \in Y^*$, we denote by $x^{**} \otimes y^*$ the linear functional defined on the space of operators by $(x^{**} \otimes y^*)(T) = x^{**}(T^*(y^*))$, for any operator $T$ and note that $\|x^{**} \otimes y^*\| = \|x^{**}\|\|y^*\|$. We denote a rank one operator by $x^* \otimes y$, so that $(x^{**} \otimes y^*)(x^* \otimes y) = x^{**}(x^*)y^*(y)$. We recall from [5] that for the space of compact operators $\mathcal{K}(X,Y)$, $\partial_e \mathcal{K}(X,Y)_1^* = \{x^{**} \otimes y^* : x^{**} \in \partial_e X_1^{**},\ y^* \in \partial_e Y_1^*\}$.

We need a lemma which is perhaps part of the folklore. Let $X = M \bigoplus_\infty N$ ($\ell^\infty$-direct sum) for closed subspaces $M, N \subset X$. We note that $X^{**} = M^{**} \bigoplus_\infty N^{**}$.

**Lemma 1.** *Let $X = M \bigoplus_\infty N$ . Let $x \in S_X$ and $x = m+n$ for $m \in M$, $n \in N$. If $\|m\| = 1 = \|n\|$ then $x$ is not a smooth point. If $\|m\| < 1$ then $x$ is a smooth point (very smooth point) of $X$ if and only if $n$ is a smooth point (very smooth point) of $N$.*

**Theorem 2.** *Let $X$ be an $L$-space and $Y$ a Banach space. Let $T \in \mathcal{L}(X,Y)$ be a smooth point such that $T$ attains its norm and only at $\pm x_0$ for some $x_0 \in \partial_e X_1$. Then $T(x_0)$ is a smooth point of $Y$ and if $P : X \to span\{x_0\}$ is the band projection, then $\|T \circ (I-P)\| < \|T\|$. Conversely suppose $T$ attains its norm only at $\pm x_0 \in \partial_e X_1$ and $T(x_0)$ is a smooth point of $Y$. If $\|T \circ (I-P)\| < \|T\|$, then $T$ is a smooth point of $\mathcal{L}(X,Y)$.*

*Proof.* Suppose $\|T\| = 1$, $T$ is a smooth point and $\|T(x_0)\| = 1$. Let $y^* \in \partial_e Y_1^*$ be such that $y^*(T(x_0)) = 1$. Since $(x_0 \otimes y^*)(T) = 1$, as $T$ is a smooth point, it is easy to see that $T(x_0)$ is a smooth point and $x_0 \in \partial_e X_1$. Since $X$ is an $L$-space, by Kakutani's representation theorem of

$X$ as $L^1(\mu)$ for a positive measure $\mu$, $x_0$ corresponds to a normalized measure atom (see Section 15 of Chapter 5 in [4]). Thus there is a projection $P : X \to span\{x_0\}$ such that $\|P(x)\| + \|x - P(x)\| = \|x\|$ for all $x \in X$. By taking adjoints, it is easy to see that $\|S\| = max\{\|S \circ P\|, \|S \circ (I - P)\|\}$ for all $S \in \mathcal{L}(X, Y)$. In particular since $\|T(P(x_0))\| = 1$ we have $\|T \circ P\| = 1$. As $T$ is a smooth point by Lemma 1 we get that $\|T \circ (I - P)\| < 1$.

Conversely suppose that $T$ attains its norm only at $\pm x_0 \in \partial_e X_1$ and $T(x_0)$ is a smooth point of $Y$, $\|T \circ (I - P)\| < 1$. Since $T(x_0)$ is a smooth point, it is easy to see that $T \circ P$ is a smooth point of $\{S \circ P : S \in \mathcal{L}(X, Y)\}$. Thus again by Lemma 1 we get that $T$ is a smooth point of $\mathcal{L}(X, Y)$. $\square$

To get a complete analogue of Abatzoglou's result, mild approximation theoretic assumptions some times can be used to achieve norm attainment. We are able to do it only for a very smooth point $T$. The following theorem illustrates this. The conditions assumed here are satisfied by an $L$-space (see [4] Chapter 5). We note that $x \in S_X$ is smooth point if and only if there is a unique $x^* \in \partial_e X_1^*$ such that $x^*(x) = 1$. It is a very smooth point if and only if $x^*$ has a unique norm preserving extension in $X^{***}$. This is equivalent to $x^*$ being a point of weak$^*$-weak continuity for the identity map on $X_1^*$. See Lemma III.2.14 in [2].

**Theorem 3.** *Let $X$ be a Banach space. Suppose there is a net $\{T_\alpha\}_{\alpha\in\Delta} \subset \mathcal{K}(X)_1$ such that both $T_\alpha$ and its adjoint converge point-wise to the identity operator. If $T \in \mathcal{L}(X,Y)$ is a very smooth point, then $T$ attains its norm at $x_0 \in \partial_e X_1$ and $T(x_0)$ is a very smooth point of $Y$.*

*Proof.* Following the ideas from [3] we will define a linear contractive projection $P : \mathcal{L}(X,Y)^* \to \mathcal{L}(X,Y)^*$ such that $\ker(P) = \mathcal{K}(X,Y)^\perp$. It is easy to see that for any $\Lambda \in \mathcal{L}(X,Y)^*$, $P(\Lambda)$ is a norm preserving extension of $\Lambda|\mathcal{K}(X,Y)$ and hence $\mathcal{K}(X,Y)^*$ is isometric to the range of $P$. By going through a subnet if necessary, we assume that $T_\alpha \to \phi$ in the weak$^*$-topology, for some $\phi \in \mathcal{K}(X)_1^{**}$. For $S \in \mathcal{L}(X,Y)$, $\Lambda \in \mathcal{L}(X,Y)^*$, let $S\Lambda : \mathcal{K}(X) \to \mathbb{R}$ be the continuous functional $S\Lambda(K) = \Lambda(SK)$, for $K \in \mathcal{K}(X)$. Let $P(\Lambda)(S) = \lim_\alpha(T_\alpha)(S\Lambda) = \phi(S\Lambda)$. It is easy to see that $P$ is a linear contraction and $\ker(P) = \mathcal{K}(X,Y)^\perp$. We next verify that for $x^{**} \in X^{**}, y^* \in Y^*$, $P(x^{**} \otimes y^*) = x^{**} \otimes y^*$. Let $S \in \mathcal{L}(X,Y)$, with the above notation, we note that $S\, x^{**} \otimes y^* = x^{**} \otimes S^*(y^*)$. Thus $P(x^{**} \otimes y^*)(S) = \lim_\alpha(x^{**} \otimes S^*(y^*))(T_\alpha) = \lim_\alpha x^{**}(T_\alpha^*(S^*(y^*))$. Since $T_\alpha^*(S^*(y^*)) \to S^*(y^*)$, we get $P(x^{**} \otimes y^*)(S) = (x^{**} \otimes y^*)(S)$. For any $S \in \mathcal{L}(X,Y)$, we have $\|S\| = sup\{|(x^{**} \otimes y^*)(S)| : x^{**} \in X_1^{**}\ ,\ y^* \in Y_1^*\}$. By a standard separation theorem argument, we get $\mathcal{L}(X,Y)_1^* = \overline{(P(\mathcal{L}(X,Y)^*)_1)}$, where the closure is taken in the weak$^*$-topology. Now since $T$ is a very smooth point, let $\Lambda \in \partial_e \mathcal{L}(X,Y)_1^*$ be such that

$\Lambda(T) = \|T\|$. Since $\Lambda$ is also a point of weak*-weak continuity for the identity map, we see that $\Lambda \in P(\mathcal{L}(X,Y)^*)_1$ (this set under the canonical isometry is $\mathcal{K}(X,Y)^*_1$, a norm and hence weakly closed convex set). As $\Lambda$ is an extreme point, we get by the result of Ruess and Stegall ([5]) that $\Lambda = x^{**} \otimes y^*$, for some $x^{**} \in \partial_e X^{**}$ and $y^* \in \partial_e Y^*_1$. Thus $\|T^*\| = \|T^*(y^*)\|$. We next claim that $x^{**}$ is a point of weak*-weak continuity for the identity map on $X_1^{**}$. Since $X_1$ is weak*-dense in $X_1^{**}$, this in particular shows that $x^{**} = x_0 \in \partial_e X_1$ and thus $T$ attains its norm. Let $\{x_\alpha^{**}\} \in X_1^{**}$ be a net such that $x^{**} \to x^{**}$ in the weak*-topology. For any $S \in \mathcal{L}(X,Y)$, we have $(x_\alpha^{**} \otimes y^*)(S) = x_\alpha^{**}((S^*)(y^*)) \to x^{**}((S^*(y^*)) = (x^{**} \otimes y^*)(S)$. Thus the net $x_\alpha^{**} \otimes y^* \in \mathcal{L}(X,Y)^*_1$ converges in the weak*-topology to $x^{**} \otimes y^*$. Therefore by weak*-weak continuity, we get that this convergence is also in the weak topology. We recall from Chapter VI of [2] that the injective tensor product, $X^* \hat{\otimes}_\epsilon Y \subset \mathcal{L}(X,Y)$ and thus by duality $(X^* \hat{\otimes}_\epsilon Y)^{**} \subset \mathcal{L}(X,Y)^{**}$. It is easy to see that $X^{***} \hat{\otimes}_\epsilon Y \subset (X^* \hat{\otimes}_\epsilon Y)^{**}$ such that under the canonical embedding, $(\tau \otimes y)(x^{**} \otimes y^*) = \tau(x^{**}) y^*(y)$, for $\tau \in X^{***}, y \in Y$. See page 265 of [2]. This implies that $x_\alpha^{**} \to x^{**}$ in the weak topology. Thus $x^{**}$ is a point of weak*-weak continuity. Similarly one can show that $y^*$ is a point of weak*-weak continuity for the identity map on $Y_1^*$. By using Lemma III.2.14 of [2] again, we get that $y^*$ has unique norm preserving extension in $Y^{***}$ so that $T(x_0)$ is a very smooth point. $\square$

One can now use the above theorem and Lemma 1 to prove the following complete analogue of Abatzoglou's result for very smooth points of $\mathcal{L}(X,Y)$.

**Theorem 4.** *Let $X$ be an $L$-space and $Y$ a Banach space. Let $T \in \mathcal{L}(X,Y)$ be a very smooth point. Then $T$ attains its norm only at $\pm x_0$ for some $x_0 \in \partial_e X_1$. $T(x_0)$ is a very smooth point of $Y$ and if $P : X \to span\{x_0\}$ is the band projection, then $\|T \circ (I-P)\| < \|T\|$. Conversely suppose $T$ attains its norm only at $\pm x_0 \in \partial_e X_1$ and $T(x_0)$ is a very smooth point of $Y$. If $\|T \circ (I-P)\| < \|T\|$, then $T$ is a very smooth point of $\mathcal{L}(X,Y)$.*

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