Linear Hahn Banach Type Extensions in Banach and Hilbert Modules

OTOA 2016
ISI BANGALORE

SUDESHNA BASU
GEORGE WASHINGTON UNIVERSITY

Hahn Banach Extension Operator

- Let X be a Banach Space, X^* its dual, M a closed subspace of X, . For each bounded linear functional f:M → R we define
- H_M(f) = { $f^{\sim} \in X^*$: || $f^{\sim}$|| = ||f|| , $f^{\sim}{}_{|M}$ =f}
- Clearly by Hahn Banach Theorem, H_M(f) is non empty.
- A selector T: $M^* \to X^*$ with Tf ∈ H_M(f) for all f ∈ M^*, is called a Hahn Banach Extension operator for M. Clearly T is norm preserving.
- If T is linear then it is called a linear Hahn Banach Extension operator .

History

J. Lindenstrauss studied this notion in the context of non separable reflexive Banach spaces.(1966)

Extensive studies on Linear Hahn-Banach extension operators by Heinrich and Mankiewicz (1982.)

Subsequently, Sims and Yost proved the existence of linear Hahn Banach extension operators via "interspersing" subspaces in a purely Banach space theoretic set up.(1989)

“Interspersing” Subspace

Theorem (Sims and Yost) Let N be a subspace of a Banach space X. Then there exists a subspace M$\supseteq$ N with dens M$=$ dens N and a **linear** Hahn Banach Extension operator T:$M^* \to X^*$.

We call this subspace an interspersing subspace of X .

Linearity of Hahn Banch extension Operator

- It is not difficult to show that if X is a Hilbert space, then T is linear .Conversely, if every subspace of X admits a linear Hahn Banach Extension operator, then X is a Hilbert space.

- So for a Banach space linearity is not available in general but due to the result by Sims and Yost

we can expect plenty of subspaces which admit linear Hahn Banach Extension operator.

Banach and Hilbert Modules, Some Notations

- Let **A** denote a Banach algebra.
- Let **X** denote a Left Banach A-module
- **X',** its A – dual which is a right A-module,

i.e. the space of all bounded module homomorphisms from X into A.

- **B(X)** will denote the space of all bounded linear operators from X to itself.
- **L(X)** space of all bounded A-linear operators from X to itself
- If X is a Left Hilbert module, then **A(X)** will denote all those bounded operators $T \in B(X)$ which have an adjoint, T^*. It turns out that $A(X) \subseteq L(X)$.

Hahn Banach Type extension Operator

Let X be a Banach A-module and X' its A-dual.

- Suppose $M \subseteq X$ is a Banach A-submodule of X. Suppose a bounded module homomorphism f from M to A has a norm preserving A-linear extension $f^{\sim}$, on the whole of X. We call $f^{\sim}$, a **Hahn Banach Type Extension** of f.

- Suppose for all $f \in M'$, there is a Hahn Banach type extension. By the Hahn Banach extension set of f, we mean the set, $H_M(f) = \{f^{\sim} \in X' : \|f^{\sim}\| = \|f\|, f^{\sim}_{|M} = f\}$. Consider $T : M' \to X'$ such that $T(f) \in H_M((f)$. Then T is called a **Hahn-Banach Type Extension operator.**

- If T is right A-linear, we call it **A-linear Hahn-Banach Type Extension operator**

Question

- Can we have Similar result like Sims And Yost

in the Banach Module set up?

i.e. Let N be a submodule of a Banach module X.

$\Rightarrow$?

There exists an interspersing submodule $M \supseteq N$ with dens $M =$ dens N and an A- linear Hahn Banach Type Extension operator T:$M' \to X'$.

Non existence of Hahn Banach Type extension

- Let A= C[0,1] denote the C*-algebra of all complex continuous functions on [0,1].

A is a Hilbert module over itself.

M = $C_0[0,1]$, the subset of all continuous functions vanishing at 0 is a Hilbert A-submodule of A.

We define a bounded A-linear map φ: M$\rightarrow$A , by φ(f)(t)= f(t)sin(1/t), f $\in$ M, t $\in$ (0,1], φ(f)(0)=0. Clearly φ cannot be extended to A.

Existence of Hahn Banach Type Extensions in Banach Modules

- Hahn Banach Type extensions may or may not exist in the context of Banach Modules

We give two examples ,the first one where it exists and the second one where it does not exist.

Let A be the C*-algebra of complex continuous functions on $[0,1] \cup [2,3]$. Then A is a Hilbert module over itself.

Let $M = \{f \in A : f = 0 \text{ on } [2,3]\}$. Then M is a Hilbert A-submodule of A. Let $g = \chi_{[0,1]} \in M$. For any $\varphi \in M'$ we define $\psi : A \to A$

as $\psi(u) = \varphi(gu)$. Clearly ψ is a Hahn-Banach Type Extension of φ .So we can now define $T : M' \to A'$ as an A-linear Hahn Banach Type Extension operator .

Hilbert Modules

Theorem :Let A be a Von Neumann algebra and X, a Hilbert A-module. Then every $\varphi : X \longrightarrow A$ has a unique Hahn Banach Type Extension, $\psi : X' \longrightarrow A$ and there exists a Hahn-Banach Type Extension operator T from X' to X'' given by $T(\varphi) = \psi$.

Follows from a result of Paschke's , X' is self dual in this case .

Multipliers of the Algebra

- Rephrasing Frank 's Theorem in this set up.

Theorem : For a C*-algebra, the following are equivalent

i) For any Hilbert module X and any submodule M, and any f: M→ A, there exists a Hahn Banach Type extension $f^{\sim}: X \to A$, such that $f^{\sim}{}_{|M^{\perp}}$ = 0

ii)For any Hilbert module X and any submodule M, there exists a A-Linear Hahn-Banach type extension operator T : M'→X'

iii) The multiplier C*algebra M(A) is monotone complete

iv) The multiplier C*algebra M(A) is additively complete

Generalised Hahn Banach Type extension Operators

Let X, Y be Banach A-modules, M,N be submodules of X and Y respectively .

- Suppose a A linear operator f from M to N has a norm preserving A-linear extension $f^{\sim}$, from X to Y . We call $f^{\sim}$, a **Generalised Hahn Banach Type Extension** of f.

- Suppose for all $f \in L(M,N)$, there is a Generalised Hahn Banach type extension. By the Hahn Banach extension set, we mean the set H_M(f) = $\{f^{\sim} \in \mathrm{L}(\mathrm{X}, \mathrm{Y})$: $||f^{\sim}|| = ||f||$, $f^{\sim}{}_{|M} = f$ }. Then, consider T : L(M,N)$\rightarrow$ L(X,Y) such that T(f) $\in H_M$((f) Then T is called a **Generalised Hahn-Banach Type Extension operator.**

- If Y is a right module as well, and T right A-linear, we call it a **A-linear Generalised Hahn-Banach Type Extension operator**

Theorem Let A be a Von Neumann algebra.

and X a Hilbert A module, then

there exists a Generalised Hahn-Banach Type Extension operator T from L(X) to L(X').

Corollary Let A be a Von Neumann algebra.

and X a Hilbert A module, then

there exists a Generalised Hahn-Banach Type Extension operator T from A(X) to A(X').

Further, T is a *-isometric isomorphism in particular, T(F^*)= $T(F)^*$, for all F ∈ A(X).

Sacrificing norm preserving extensions

Let X be a Banach A-module and X' its A-dual.

- Suppose M⊆ X is a Banach A-submodule of X. Suppose a bounded module homomorphism f from M to A has a bounded A-linear extension $f^{\sim}$, on the whole of X. We call $f^{\sim}$, a **Weak Hahn Banach Type Extension** of f.

- Suppose for all f ∈ M', there is a weak Hahn Banach type extension. By the Hahn Banach extension set, we mean the set , WH_M(f) = {$f^{\sim}$ ∈ X' : , $f^{\sim}{}_{|M}$= f }. Then, consider T : M'→X' such that T(f) ∈ WH_M((f). Then T is called a **Weak Hahn-Banach Type Extension operator.**

- If T is A-linear, we call it **A-linear weak Hahn-Banach Type Extension operator**

Weak Hahn Banach type operators

An observation from Frank again

Theorem Let A be a C*-algebra, X be an A-reflexive Hilbert A-module, i.e. X = X''. Then for Hilbert A-submodules M ⊆ X the following two conditions are equivalent:

(i) M is a topological direct summand of X (i.e. not necessarily an orthogonal direct summand).

(ii) M ≡ $M^{\perp\perp}$ ⊆ X and there exists a bounded A-linear weak Hahn Banach type extension operator from M' to X'.

Example of Weak HB type operator

Consider the C*-algebra A = l_∞, its two-sided ideal I = c_0 and

the Hilbert A-module X = A ⊕ I equipped with the A-valued inner product

h(a, i), (b, j)i = ab∗ + ij∗ for (a, i), (b, j) ∈ X. The Hilbert A-submodule

M = {(i, i) : i ∈ I} is a topological direct summand of X since X can be decomposed

as X = M+ {(a, 0) : a ∈ A}, but it is not an orthogonal direct summand.

Moreover, M ≡ $M^{\perp\perp}$. Then there exists a weak Hahn Banach type extension operator from M′ to X′ = = A⊕A.

Clearly, A-dual Hilbert A-module M′ = {(a, a) : a ∈ A} can

be boundedly and A-linearly embedded into the Hilbert A-module X′ = A⊕A

the rule ϕ((a, a))(b, i) := 2ia∗ for a, b ∈ A, i ∈ I. (But the restriction of ϕ(a,a) ∈ X′

to $M^{\perp}$ = {(−i, i) : i ∈ I} ⊂ X is obviously non-zero.)

So M can work as an interspersing submodule of X

Examples of interspersing submodules

X:Banach Space,

B(X): Bounded liner operators from X to X.

For x ∈X, ||x||=1, there exists $\varphi_x \in X^*$

such that || φ_x || = ||x||=1.

Define

M_x ={ T ∈ B(X) : T(x)=0}, N_x= {$\varphi_x \otimes$y : y ∈X}

They are both left ideals. It turns out that for M_x and N_x there exist Generalised Hahn Banach Type extension operators from L(M_x, B(X)) to L(B(X)) and L(N_x, B(X)) to L(B(X)).

M_x can act as an interspersing submodule.

(Dales, et al,2013) Dichotomy of Maximal left Ideals

Let X be a non-zero Banach space. Then, for each maximal left ideal L of B(X), exactly one

of the following two alternatives holds:

- (i) L =M_x for some x ϵ X, ||x||=1.
- (ii) L contains F(X) where F(X) is the two sided

ideal of finite rank operators.

We have already observed that M_x can act as an interspersing submodule. For closed left ideals containing F(X) the situation is more complicated.