Composition operators on some analytic reproducing kernel Hilbert spaces

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Operators

- By an **operator** in a complex Hilbert space $\mathcal{H}$ we mean a linear mapping $A: \mathcal{H} \supseteq \mathcal{D}(A) \to \mathcal{H}$ defined on a vector subspace $\mathcal{D}(A)$ of $\mathcal{H}$, called the **domain** of A;
- We say that a densely defined operator A in $\mathcal{H}$ is
  - *positive* if $\langle A\xi, \xi\rangle \geqslant 0$ for all $\xi \in \mathcal{D}(A)$; then we write $A \geqslant 0$,
  - *selfadjoint* if $A = A^*$,
  - *hyponormal* if $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|A^*\xi\| \leqslant \|A\xi\|$ for all $\xi \in \mathcal{D}(A)$,
  - *cohyponormal* if $\mathcal{D}(A^*) \subseteq \mathcal{D}(A)$ and $\|A\xi\| \leqslant \|A^*\xi\|$ for all $\xi \in \mathcal{D}(A^*)$,
  - *normal* if A is hyponormal and cohyponormal,
  - *subnormal* if there exist a complex Hilbert space $\mathcal{M}$ and a normal operator N in $\mathcal{M}$ such that $\mathcal{H} \subseteq \mathcal{M}$ (isometric embedding), $\mathcal{D}(A) \subseteq \mathcal{D}(N)$ and $Af = Nf$ for all $f \in \mathcal{D}(A)$,
  - *seminormal* if A is either hyponormal or cohyponormal.

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  - *seminormal* if A is either hyponormal or cohyponormal.

The class $\mathscr{F}$

- $\mathscr{F}$ stands for the class of all entire functions Φ of the form

$$\Phi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C}, \tag{1}$$

  such that $a_k \geqslant 0$ for all $k \geqslant 0$ and $a_n > 0$ for some $n \geqslant 1$.
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The group $\mathfrak{G}_\Phi$

- If $\Phi \in \mathscr{F}$ is as in (1), we set

$$\mathscr{Z}_\Phi = \{n \in \mathbb{N} \colon a_n > 0\}$$

and define the multiplicative group $\mathfrak{G}_\Phi$ by

$$\mathfrak{G}_\Phi = \bigcap_{n \in \mathscr{Z}_\Phi} G_n,$$

where G_n is the multiplicative group of nth roots of 1, i.e.,

$$G_n := \{z \in \mathbb{C} \colon z^n = 1\}, \quad n \geqslant 1.$$

The order of the group $\mathfrak{G}_\Phi$ can be calculated explicitly.

The RKHS $\Phi(\mathcal{H})$

- $\mathcal{H}$ is a complex Hilbert space with inner product $\langle\cdot, \text{-}\rangle$. If $\Phi \in \mathscr{F}$, then by the Schur product theorem, the kernel $K^\Phi\colon \mathcal{H}\times\mathcal{H}\to\mathbb{C}$ defined by

$$K^\Phi(\xi,\eta) = K^{\Phi,\mathcal{H}}(\xi,\eta) = \Phi(\langle\xi,\eta\rangle), \quad \xi,\eta\in\mathcal{H},$$

  is positive definite.
- $\Phi(\mathcal{H})$ stands the reproducing kernel Hilbert space with the reproducing kernel K^Φ; $\Phi(\mathcal{H})$ consists of holomorphic functions on $\mathcal{H}$.
- Reproducing property of $\Phi(\mathcal{H})$:

$$f(\xi) = \langle f, K^\Phi_\xi\rangle, \quad \xi\in\mathcal{H},\ f\in\Phi(\mathcal{H}),$$

  where

$$K^\Phi_\xi(\eta) = K^{\Phi,\mathcal{H}}_\xi(\eta) = K^\Phi(\eta,\xi), \quad \xi,\eta\in\mathcal{H}.$$

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Some examples - I

- Frankfurt spaces [1975/6/7];
  Multidimensional generalizations - Szafraniec [2003].
- For ν, a positive Borel measure on $\mathbb{R}_+$ such that

$$\int_{\mathbb{R}_+} t^n \,\mathrm{d}\,\nu(t) < \infty \text{ and } \nu((c,\infty)) > 0 \text{ for all } n \in \mathbb{Z}_+ \text{ and } c > 0.$$

  we define the positive Borel measure μ on $\mathbb{C}$ by

$$\mu(\varDelta) = \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}_+} \chi_\varDelta(r\,\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\,\nu(r)\,\mathrm{d}\,\theta, \quad \varDelta \text{ - Borel subset of } \mathbb{C}.$$

- Then we define the function $\varPhi \in \mathscr{F}$ by

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Some examples - II

- Frankfurt proved that $\Phi(\mathbb{C})$ can be described as follows

$$\Phi(\mathbb{C}) = \Big\{f\colon f \text{ - entire function } \& \ f \in L^2(\mu)\Big\}; \tag{2}$$

- hence the right-hand side of (2) is a reproducing kernel Hilbert space with the reproducing kernel

$$\mathbb{C} \times \mathbb{C} \ni (\xi, \eta) \longmapsto \sum_{n=0}^{\infty} \frac{1}{\int_{\mathbb{R}_+} t^{2n} \,\mathrm{d}\,\nu(t)} \xi^n \bar{\eta}^n \in \mathbb{C}.$$

- If $\int_{\mathbb{R}_+} t^{2n} \,\mathrm{d}\,\nu(t) = n!$ for all $n \in \mathbb{Z}_+$, then $\Phi = \exp$, μ is the Gaussian measure on $\mathbb{C}$ and $\Phi(\mathbb{C})$ is the Segal-Bargmann space $\mathcal{B}_1$ of order 1.

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- Given a holomorphic mapping $\varphi\colon \mathcal{H} \to \mathcal{H}$, we define the operator C_φ in $\Phi(\mathcal{H})$, called a *composition operator* with a *symbol* φ, by

$$\mathcal{D}(C_\varphi) = \{f \in \Phi(\mathcal{H})\colon f \circ \varphi \in \Phi(\mathcal{H})\},$$
$$C_\varphi f = f \circ \varphi, \quad f \in \mathcal{D}(C_\varphi).$$

- C_φ is always closed.
- If $\Phi(0) \neq 0$ and $C_\varphi \in \boldsymbol{B}(\Phi(\mathcal{H}))$, then $r(C_\varphi) \geqslant 1$ and thus $\|C_\varphi\| \geqslant 1$.

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Maximality of C_ϕ

Theorem

Let $\varPhi \in \mathscr{F}$ and $\varphi, \psi \colon \mathcal{H} \to \mathcal{H}$ be holomorphic mappings. Assume that the operators C_φ and C_ψ are densely defined in $\varPhi(\mathcal{H})$. Then the following conditions are equivalent:

1. $C_\varphi \subseteq C_\psi$,
2. $C_\varphi = C_\psi$,
3. *there exists $\alpha \in \mathfrak{G}_\varPhi$ such that $\varphi(\xi) = \alpha \cdot \psi(\xi)$ for every $\xi \in \mathcal{H}$.*

Proposition

Suppose $\Phi \in \mathscr{F}$, $\varphi\colon \mathcal{H} \to \mathcal{H}$ *is a holomorphic mapping and* $\mathcal{D}(C_\varphi) = \Phi(\mathcal{H})$. *Then* C_φ *is bounded and there exists a unique pair* $(A, b) \in \boldsymbol{B}(\mathcal{H}) \times \mathcal{H}$ *such that* $\varphi = A + b$, *i.e.,*
$\varphi(\xi) = A\xi + b, \quad \xi \in \mathcal{H}.$

The Segal-Bargmann space over $\mathbb{C}^d$ [B. J. Carswell, B. D. MacCluer, A. Schuster 2003]

Fock's type model for C_A

Theorem

Suppose $\Phi \in \mathscr{F}$, Q is a conjugation on $\mathcal{H}$ and $A \in \boldsymbol{B}(\mathcal{H})$. Then there exists a unitary isomorphism $U = U_{\Phi,Q} \colon \Phi(\mathcal{H}) \to \bigoplus_{n \in \mathscr{Z}_\Phi} \mathcal{H}^{\odot n}$ such that

$$C_A^* = U^{-1} \varGamma_\Phi(\varXi_Q(A)) U,$$

where $\varXi_Q(A) = QAQ$, $\varGamma_\Phi(T) = \bigoplus_{n \in \mathscr{Z}_\Phi} T^{\odot n}$ and $T^{\odot n}$ is the nth symmetric tensor power of $T \in \boldsymbol{B}(\mathcal{H})$.

Theorem

Suppose $\Phi \in \mathscr{F}$ *and* $A \in \boldsymbol{B}(\mathcal{H})$. *Then*

(i) $C_A^* = C_{A^*}$,

(ii) $\mathscr{K}^{\Phi}$ *is a core for* C_A.

Boundedness of C_A

Theorem

Suppose $\varPhi \in \mathscr{F}$ and $A \in \boldsymbol{B}(\mathcal{H})$. Set[a] *$m = \min \mathscr{Z}_\varPhi$ and $n = \sup \mathscr{Z}_\varPhi$. Then*

1. *if $n < \infty$, then $C_A \in \boldsymbol{B}(\varPhi(\mathcal{H}))$,*
2. *if $n = \infty$, then $C_A \in \boldsymbol{B}(\varPhi(\mathcal{H}))$ if and only if $\|A\| \leqslant 1$.*
3. *Moreover, if $C_A \in \boldsymbol{B}(\varPhi(\mathcal{H}))$, then $\|C_A\| = q_{m,n}(\|A\|)$ and $r(C_A) = q_{m,n}(r(A))$.*

[a] Note that 0 is a zero of $\varPhi$ of multiplicity m and ∞ is a pole of $\varPhi$ of order n.

- If $m \in \mathbb{Z}_+$ and $n \in \mathbb{Z}_+ \cup \{\infty\}$, then

$$q_{m,n}(\vartheta) = \vartheta^m \max\{1, \vartheta^{n-m}\}, \quad \vartheta \in [0, \infty),$$

where $\vartheta^0 = 1$ for $\vartheta \in [0,\infty)$, $\vartheta^\infty = \infty$ for $\vartheta \in (1,\infty)$, $\vartheta^\infty = 0$ for $\vartheta \in [0,1)$ and $1^\infty = 1$.

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1. *if* $n < \infty$, *then* $C_A \in \boldsymbol{B}(\varPhi(\mathcal{H}))$,
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When is C_A an isometry, ..., a partial isometry?

Proposition

Suppose $\varPhi \in \mathscr{F}$ and $A \in \boldsymbol{B}(\mathcal{H})$. Then C_A is an isometry (resp.: a coisometry, a unitary operator) if and only if A is a coisometry (resp.: an isometry, a unitary operator).

Proposition

Let $\varPhi \in \mathscr{F}$ and $P \in \boldsymbol{B}(\mathcal{H})$. Then C_P is an orthogonal projection if and only if there exists $\alpha \in \mathfrak{G}_\varPhi$ such that αP is an orthogonal projection.

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Positivity of C_A

Theorem

Suppose $\varPhi \in \mathscr{F}$ and $A \in \boldsymbol{B}(\mathcal{H})$. Then the following conditions are equivalent:

1. $C_A \geqslant 0$,
2. *there exists* $\alpha \in \mathfrak{G}_\varPhi$ *such that* $\alpha A \geqslant 0$,
3. *there exists* $B \in \boldsymbol{B}(\mathcal{H})$ *such that* $B \geqslant 0$ *and* $C_A = C_B$.
4. *Moreover, if* $A \geqslant 0$, *then* C_A *is selfadjoint and* $C_A = C^*_{A^{1/2}} C_{A^{1/2}}$.

Powers of a positive C_A

Theorem

Let $\Phi \in \mathscr{F}$, $A \in \boldsymbol{B}(\mathcal{H})$ and $t \in (0, \infty)$. Suppose $A \geqslant 0$. Then

1. C_A *is selfadjoint and* $C_A \geqslant 0$,
2. $C_A^t = C_{A^t}$,
3. $\mathcal{D}(C_{A^t}) \subseteq \mathcal{D}(C_{A^s})$ *for every* $s \in (0, t)$.

Theorem

Suppose that $\varPhi \in \mathscr{F}$ and $A \in \boldsymbol{B}(\mathcal{H})$. Let $A = U|A|$ be the polar decomposition of A. Then $C_A = C_U C_{|A^*|}$ is the polar decomposition of C_A. In particular, $|C_A| = C_{|A^*|}$.

Seminormality of C_A

Theorem

If $\Phi \in \mathscr{F}$ and $A, B \in \boldsymbol{B}(\mathcal{H})$, then the following conditions are equivalent:

1. $\mathcal{D}(C_B) \subseteq \mathcal{D}(C_A)$ *and* $\|C_A f\| \leqslant \|C_B f\|$ *for all* $f \in \mathcal{D}(C_B)$,
2. $\|C_A f\| \leqslant \|C_B f\|$ *for all* $f \in \mathscr{K}^{\Phi}$,
3. $\|A^*\xi\| \leqslant \|B^*\xi\|$ *for all* $\xi \in \mathcal{H}$.

Theorem

If $\Phi \in \mathscr{F}$ and $A \in \boldsymbol{B}(\mathcal{H})$, then the following conditions are equivalent:

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Inequalities

Theorem

Let $\Phi \in \mathscr{F}$ and let $A, B \in \boldsymbol{B}_+(\mathcal{H})$. Then the following conditions are equivalent:

1. $C_A \preccurlyeq C_B$,
2. $\langle C_A f, f\rangle \leqslant \langle C_B f, f\rangle$ for all $f \in \mathscr{K}^{\Phi}$,
3. $A \leqslant B$.

Generalized inverses

- Suppose $A \in \boldsymbol{B}(\mathcal{H})$ is selfadjoint. It is well-known (and easy to verify) that $A|_{\overline{\mathcal{R}(A)}} \colon \overline{\mathcal{R}(A)} \to \mathcal{R}(A)$ is a bijection.
- Hence, we may define a generalized inverse A^{-1} of A by

$$A^{-1} = (A|_{\overline{\mathcal{R}(A)}})^{-1}.$$

- A^{-1} is an operator in $\mathcal{H}$ (not necessarily densely defined) such that

$$\mathcal{D}(A^{-1}) = \mathcal{R}(A), \quad \mathcal{R}(A^{-1}) = \overline{\mathcal{R}(A)},$$
$$AA^{-1} = I_{\mathcal{R}(A)} \quad \text{and} \quad A^{-1}A = P,$$

where $I_{\mathcal{R}(A)}$ is the identity operator on $\mathcal{R}(A)$ and P is the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathcal{R}(A)}$.
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Generalized inverses

- Suppose $A \in \boldsymbol{B}(\mathcal{H})$ is selfadjoint. It is well-known (and easy to verify) that $A|_{\overline{\mathcal{R}(A)}}\colon \overline{\mathcal{R}(A)} \to \mathcal{R}(A)$ is a bijection.
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- If $\mathcal{R}(A) = \mathcal{R}(B) = \mathcal{H}$ ($\iff$ $A^{-1}, B^{-1} \in \boldsymbol{B}(\mathcal{H})$), then $B^{-1} \preccurlyeq A^{-1}$ if and only if $B^{-1} \leqslant A^{-1}$ (i.e., $\langle B^{-1}f, f\rangle \leqslant \langle A^{-1}f, f\rangle$ for all $f \in \mathcal{H}$.)

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Ranges of WOT limits

Lemma

Assume $\{A_P\}_{P\in\mathcal{P}} \subseteq \boldsymbol{B}_+(\mathcal{H})$ is a monotonically decreasing net which converges in WOT to $A \in \boldsymbol{B}_+(\mathcal{H})$. If $\xi \in \mathcal{H}$, then TFAE:

(i) $\xi \in \mathcal{R}(A^{1/2})$,

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Moreover, if $\xi \in \mathcal{R}(A^{1/2})$, then $c = \|A^{-1/2}\xi\|$.

Apply

Theorem (Mac Nerney-Shmul'yan theorem)

If $A \in \boldsymbol{B}_+(\mathcal{H})$ and $\xi \in \mathcal{H}$, then TFAE:

(i) $\xi \in \mathcal{R}(A^{1/2})$,

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Boundedness of C_φ in $\exp(\mathcal{H})$

Theorem (main)

Let $\Phi = \exp$, $\varphi\colon \mathcal{H} \to \mathcal{H}$ *be a holomorphic mapping and* $\mathcal{P} \subseteq \boldsymbol{B}(\mathcal{H})$ *be an upward-directed partially ordered set of orthogonal projections of finite rank such that* $\bigvee_{P\in\mathcal{P}} \mathcal{R}(P) = \mathcal{H}$. *Then the following conditions are equivalent*:

(i) $C_\varphi \in \boldsymbol{B}(\exp(\mathcal{H}))$,

(ii) $\varphi = A + b$, *where* $A \in \boldsymbol{B}(\mathcal{H})$, $\|A\| \leqslant 1$, $b \in \mathcal{R}(I - |A^*|P|A^*|)$ *for every* $P \in \mathcal{P}$ *and*

$$S(A, b) := \sup\{\langle (I - |A^*|P|A^*|)^{-1} b, b\rangle \colon P \in \mathcal{P}\} < \infty,$$

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- *Moreover, if* $C_\varphi \in \boldsymbol{B}(\exp(\mathcal{H}))$, *then*

$$\|C_\varphi\|^2 = \exp(\|(I - AA^*)^{-1/2} b\|^2) = \exp(S(A, b)).$$

Comments

- The case of $\mathcal{H} = \mathbb{C}^n$ was proved by **Carswell, MacCluer and Schuster** in 2003 (of course without (ii)).
- In fact, our statement differs from the above, however they are equivalent if $\dim \mathcal{H} < \infty$.
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- We begin with the following proposition.

Proposition

If $\Phi \in \mathscr{F}$, $\varphi\colon \mathcal{H} \to \mathcal{H}$ is a holomorphic mapping and $\mathcal{D}(C_\varphi) = \Phi(\mathcal{H})$, then C_φ is bounded and there exists a unique pair $(A, b) \in \boldsymbol{B}(\mathcal{H}) \times \mathcal{H}$ such that $\varphi = A + b$.

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- **An idea the proof of the Proposition.**
  Noting that for all $\xi \in \mathcal{H} \setminus \{0\}$,

$$\frac{\Phi(\|\varphi(\xi)\|^2)}{\Phi(\|\xi\|^2)} = \frac{\|K^{\Phi}_{\varphi(\xi)}\|^2}{\|K^{\Phi}_{\xi}\|^2} = \left\| C^*_{\varphi}\left(\frac{K^{\Phi}_{\xi}}{\|K^{\Phi}_{\xi}\|}\right)\right\|^2 \leqslant \|C_{\varphi}\|^2,$$

and using

Lemma (The cancellation principle)

If $\Phi \in \mathscr{F}$ and $f, g\colon \mathcal{H} \to [0, \infty)$ are such that $\liminf_{\|\xi\|\to\infty} g(\xi) > 0$ and $\limsup_{\|\xi\|\to\infty} \frac{\Phi(f(\xi))}{\Phi(g(\xi))} < \infty$, then $\limsup_{\|\xi\|\to\infty} \frac{f(\xi)}{g(\xi)} < \infty$.

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If $\Phi \in \mathscr{F}$, $\varphi = A + b$ and $\psi = |A^*| + b$ ($A \in \boldsymbol{B}(\mathcal{H})$, $b \in \mathcal{H}$), then

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Lemma (A version of (i)⇔(ii) of the main result when $A \geqslant 0$)

Suppose $A \in \boldsymbol{B}_+(\mathcal{H})$, $b \in \mathcal{H}$ and $\mathcal{P} \subseteq \boldsymbol{B}(\mathcal{H})$ is an upward-directed partially ordered set of finite rank orthogonal projections such that $\bigvee_{P \in \mathcal{P}} \mathcal{R}(P) = \mathcal{H}$. Then TFAE:

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  (i)⇒(ii) One can show [!] that $C_{A+b} \in \boldsymbol{B}(\exp(\mathcal{H}))$ implies that $\|A\| \leqslant 1$ and $b \in \mathcal{R}((I - A^2)^{1/2})$. Take $P \in \mathcal{P}$. Since $APA \leqslant A^2$, we see that $I - APA \geqslant I - A^2 \geqslant 0$. By the Douglas theorem we have

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Since C_P is an orthogonal projection [!],
$C_{AP+b} = C_P C_{A+b} \in \boldsymbol{B}(\exp(\mathcal{H}))$ and $\|C_{B+b}\| = \|C_{|B^*|+b}\|$,
one can deduce that (with $B = AP$)

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This implies that $\exp(S(A, b)) \leqslant \|C_{A+b}\|^2$.

- **(ii)⇒(i)** Take $P \in \mathcal{P}$. Using the Proposition from the previous slide, we see that $C_{AP+b} \in \boldsymbol{B}(\exp(\mathcal{H}))$, $C_{(APA)^{1/2}+b} \in \boldsymbol{B}(\exp(\mathcal{H}))$, $\|C_{AP+b}\| = \|C_{(APA)^{1/2}+b}\|$ and

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Now applying

Proposition

If $\Phi \in \mathscr{F}$ and $\mathcal{P} \subseteq \boldsymbol{B}(\mathcal{H})$ is an upward-directed partially ordered set of orthogonal projections, then

$$\lim_{P \in \mathcal{P}} C_P f = C_Q f, \quad f \in \Phi(\mathcal{H}),$$

where Q is the orthogonal projection of $\mathcal{H}$ onto $\bigvee_{P \in \mathcal{P}} \mathcal{R}(P)$.

we deduce that

$$\|C_{A+b} f\|^2 \leqslant \exp(S(A, b)) \|f\|^2, \quad f \in \mathcal{D}(C_{A+b}).$$

Since composition operators are closed and C_{A+b} is densely defined [!], this implies that $C_{A+b} \in \boldsymbol{B}(\exp(\mathcal{H}))$ and $\|C_{A+b}\|^2 \leqslant \exp(S(A, b))$, which completes the proof of the Lemma.

Now applying

Proposition

If $\Phi \in \mathscr{F}$ and $\mathcal{P} \subseteq \boldsymbol{B}(\mathcal{H})$ is an upward-directed partially ordered set of orthogonal projections, then

$$\lim_{P \in \mathcal{P}} C_P f = C_Q f, \quad f \in \Phi(\mathcal{H}),$$

where Q is the orthogonal projection of $\mathcal{H}$ onto $\bigvee_{P \in \mathcal{P}} \mathcal{R}(P)$.

we deduce that

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Sketch of the proof 7

- **The proof of (ii)⇔(iii) of the main result.**
  Without loss of generality we may assume that A is a contraction. Set $A_P = I - |A^*|P|A^*|$ for $P \in \mathcal{P}$. Then $A_P \in \boldsymbol{B}_+(\mathcal{H})$ for all $P \in \mathcal{P}$. Since $\bigvee_{P\in\mathcal{P}} \mathcal{R}(P) = \mathcal{H}$, we see that $\{P\}_{P\in\mathcal{P}}$ is a monotonically increasing net which converges in the SOT to the identity operator I.

  This implies that $\{A_P\}_{P\in\mathcal{P}} \subseteq \boldsymbol{B}_+(\mathcal{H})$ is a monotonically decreasing net which converges in the WOT to $I - |A^*|^2$. Since $\dim \mathcal{R}(|A^*|P|A^*|) < \infty$ for all $P \in \mathcal{P}$, one can show [!] that $\mathcal{R}(A_P)$ is closed and $\mathcal{R}(A_P) = \mathcal{R}(A_P^{1/2})$ for all $P \in \mathcal{P}$. Hence, by our first lemma in this presentation, $\langle A_P^{-1}\xi, \xi\rangle = \|A_P^{-1/2}\xi\|^2$ for all $\xi \in \mathcal{R}(A_P)$ and $P \in \mathcal{P}$.

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Assume $\{A_P\}_{P\in\mathcal{P}} \subseteq \boldsymbol{B}_+(\mathcal{H})$ is a monotonically decreasing net which converges in WOT to $A \in \boldsymbol{B}_+(\mathcal{H})$. If $\xi \in \mathcal{H}$, then TFAE:

(i) $\xi \in \mathcal{R}(A^{1/2})$,

(ii) *for every $P \in \mathcal{P}$, $\xi \in \mathcal{R}(A_P^{1/2})$ and $c := \sup_{P\in\mathcal{P}} \|A_P^{-1/2}\xi\| < \infty$.*

Moreover, if $\xi \in \mathcal{R}(A^{1/2})$, then $c = \|A^{-1/2}\xi\|$.

we deduce that the conditions (ii) and (iii) are equivalent and $\exp(\|(I - AA^*)^{-1/2}b\|^2) = \exp(S(A, b))$. This completes the proof of the main result.

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Another proof of CMS theorem

Theorem (Carswell, MacCluer, Schuster)

Let $\varphi\colon \mathbb{C}^d \to \mathbb{C}^d$ be a holomorphic mapping ($d \in \mathbb{N}$). Then $C_\varphi \in \boldsymbol{B}(\mathcal{B}_d)$ if and only if there exist $A \in \boldsymbol{B}(\mathbb{C}^d)$ and $b \in \mathbb{C}^d$ such that $\varphi = A + b$, $\|A\| \leqslant 1$ and $b \in \mathcal{R}(I - AA^*)$. Moreover, if $C_\varphi \in \boldsymbol{B}(\mathcal{B}_d)$, then

$$\|C_\varphi\|^2 = \exp(\langle (I - AA^*)^{-1} b, b\rangle).$$

- **Proof**

  First we reduce the proof to the case of $d = 1$ (skipped).

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- **Proof**
  First we reduce the proof to the case of $d = 1$ (skipped).

Lemma

Fix $\alpha \in [0, 1)$ and $b \in \mathbb{C}$. Let D be an operator in $\mathcal{B}_1$ given by

$$(Df)(z) = f(\alpha z + b)\exp(z\bar{b}), \quad z \in \mathbb{C}, f \in \mathcal{B}_1.$$

Then $D \in \boldsymbol{B}(\mathcal{B}_1)$ and

$$\|D\| \leqslant \frac{\exp\left(\frac{|b|^2}{1-\alpha}\right)}{\sqrt{1-\alpha^2}}.$$

- **Proof of the Lemma:**

$$
\begin{aligned}
\pi \int_{\mathbb{C}} |Df|^2 \,\mathrm{d}\,\mu_1 &= \int_{\mathbb{C}} |f(\alpha z + b)|^2 \, e^{2\mathfrak{Re}(z\bar{b})} \, e^{-|z|^2} \,\mathrm{d}\, V_1(z) \\
&\leqslant \|f\|^2 \int_{\mathbb{C}} e^{|\alpha z + b|^2 + 2\mathfrak{Re}(z\bar{b}) - |z|^2} \,\mathrm{d}\, V_1(z) \\
&= \|f\|^2 \exp\left(\frac{2|b|^2}{1-\alpha}\right) \int_{\mathbb{C}} e^{-(1-\alpha^2)\left|z - \frac{b}{1-\alpha}\right|^2} \,\mathrm{d}\, V_1(z) \\
&= \|f\|^2 \exp\left(\frac{2|b|^2}{1-\alpha}\right) \int_{\mathbb{C}} e^{-(1-\alpha^2)|z|^2} \,\mathrm{d}\, V_1(z) \\
&= \pi \|f\|^2 \frac{\exp\left(\frac{2|b|^2}{1-\alpha}\right)}{1-\alpha^2}, \quad f \in \mathcal{B}_1,
\end{aligned}
$$

Lemma

If D is as in the previous Lemma, then

$$(D^n f)(z) = f\Big(\alpha^n z + b_n\Big)\, e^{z\bar{b}_n} \exp\Big(\frac{|b|^2}{1-\alpha}\Big(n - 1 - \frac{\alpha - \alpha^n}{1-\alpha}\Big)\Big),$$

for all $z \in \mathbb{C}$, $f \in \mathcal{B}_1$ *and* $n \in \mathbb{N}$, *where* $b_n = \frac{1-\alpha^n}{1-\alpha} b$ *for* $n \in \mathbb{N}$.

- Combining the previous two Lemmata with Gelfand's formula for the spectral radius, one can prove the following.

Lemma

Let $A \in \mathbb{C}$ be such that $|A| < 1$ and let $b \in \mathbb{C}$. Set $\varphi(z) = Az + b$ for $z \in \mathbb{C}$. Then $C_\varphi \in \boldsymbol{B}(\mathcal{B}_1)$ and

$$\|C_\varphi\|^2 = \exp\left(\frac{|b|^2}{1-|A|^2}\right).$$

- If $C_{A+b} \in \boldsymbol{B}(\exp(\mathcal{H}))$, then (with $\varphi = A + b$)

$$\|C_\varphi^n\|^2 = \|C_{A^n+b_n}\|^2 = \exp(\|(I - A^n A^{*n})^{-1/2} b_n\|^2), \quad n \in \mathbb{Z}_+.$$

where $b_n = (I + \ldots + A^{n-1})b$ for $n \in \mathbb{N}$.

- The rate of growth of $\{\|(I - A^n A^{*n})^{-1/2} b_n\|\}_{n=1}^\infty$.

Proposition

Suppose $C_\varphi \in \boldsymbol{B}(\exp(\mathcal{H}))$, where $\varphi = A + b$ with $A \in \boldsymbol{B}(\mathcal{H})$ and $b \in \mathcal{H}$. Then the following holds:

(i) $\varphi^n = A^n + b_n$ *and* $b_n \in \mathcal{R}((I - A^n A^{*n})^{1/2})$ *for all* $n \in \mathbb{N}$,

(ii) *there exists a constant* $M \in (0, \infty)$ *such that*

$$\|(I - A^n A^{*n})^{-1/2} b_n\| \leqslant M\sqrt{n}, \quad n \in \mathbb{N}.$$

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- If $C_{A+b} \in \boldsymbol{B}(\exp(\mathcal{H}))$, then (with $\varphi = A + b$)

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Powers of C_{A+b}

- If $C_{A+b} \in \boldsymbol{B}(\exp(\mathcal{H}))$, then (with $\varphi = A + b$)

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Spectral radius 1

Proposition

Suppose $\varphi = A + b$, where $A \in \boldsymbol{B}(\mathcal{H})$, $b \in \mathcal{H}$ and $\|A\| < 1$. Then $C_\varphi \in \boldsymbol{B}(\exp(\mathcal{H}))$ and $r(C_\varphi) = 1$. Moreover, if $b \neq 0$, then C_φ is not normaloid.

- **Proof.**
  It follows from our main theorem that $C_\varphi \in \boldsymbol{B}(\exp(\mathcal{H}))$. Since $\|A\| < 1$, we deduce from C. Neumann's theorem that $(I - A)^{-1} \in \boldsymbol{B}(\mathcal{H})$ and

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Applying C. Neumann's theorem again, we see that $(I - A^n A^{*n})^{-1} \in \boldsymbol{B}(\mathcal{H})$ for all $n \in \mathbb{N}$ and

$$\begin{aligned} \|(I - A^n A^{*n})^{-1/2} b_n\|^2 &= \langle (I - A^n A^{*n})^{-1} b_n, b_n \rangle \\ &\leqslant \frac{\|(I - A^n)(I - A)^{-1} b\|^2}{1 - \|A\|^{2n}} \\ &\leqslant \frac{4\|b\|^2}{(1 - \|A\|^{2n})(1 - \|A\|)^2}, \quad n \in \mathbb{N}. \end{aligned}$$

This, together with Gelfand's formula for the spectral radius

$$r(C_\varphi) = \lim_{n\to\infty} \|C_\varphi^n\|^{1/n} = \lim_{n\to\infty} \exp\left(\frac{1}{2n}\|(I - A^n A^{*n})^{-1/2} b_n\|^2\right).$$

gives $r(C_\varphi) = 1$. As $\mathcal{H} \neq \{0\}$, we infer from the equality $\|C_\varphi\|^2 = \exp(\|(I - AA^*)^{-1/2} b\|^2)$ that $\|C_\varphi\| > 1$ whenever $b \neq 0$. Hence, $r(C_\varphi) \neq \|C_\varphi\|$, which means that C_φ is not normaloid.

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Theorem

If $\varphi\colon \mathbb{C}^d \to \mathbb{C}^d$ is a holomorphic mapping ($d \in \mathbb{N}$) such that $C_\varphi \in \boldsymbol{B}(\mathcal{B}_d)$, then $r(C_\varphi) = 1$.

The proof of this theorem is more subtle.

Theorem

Assume $\varphi = A + b$ with $A \in \boldsymbol{B}(\mathbb{C}^d)$ and $b \in \mathbb{C}^d$, and $C_\varphi \in \boldsymbol{B}(\mathcal{B}_d)$ ($d \in \mathbb{N}$). Then the following conditions are equivalent:

(i) *C_φ is normaloid,*

(ii) $b = 0$.

Moreover, if C_φ is normaloid, then $r(C_\varphi) = \|C_\varphi\| = 1$.

- Hence there are no bounded seminormal composition operators on the Bargmann-Segal space $\mathcal{B}_d$ of finite order d whose symbols have nontrivial translation part b.

Spectral radius: $\dim \mathcal{H} < \infty$

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Assume $\varphi = A + b$ *with* $A \in \boldsymbol{B}(\mathbb{C}^d)$ *and* $b \in \mathbb{C}^d$, *and* $C_\varphi \in \boldsymbol{B}(\mathcal{B}_d)$ $(d \in \mathbb{N})$. *Then TFAE:*

(i) C_φ *is seminormal,*

(ii) C_φ *is normal,*

(iii) A *is normal and* $b = 0$.

Example

Let $\mathcal{H}$ be an infinite dimensional Hilbert space, $V \in \boldsymbol{B}(\mathcal{H})$ be an isometry and $b \in \mathcal{H}$. Set $\varphi = V + b$. By our main theorem, we see that

$$C_\varphi \in \boldsymbol{B}(\exp(\mathcal{H})) \iff b \in \mathcal{N}(V^*).$$

Suppose V is not unitary, i.e., $\mathcal{N}(V^*) \neq \{0\}$. Take $b \in \mathcal{N}(V^*) \setminus \{0\}$. Then $\{V^n b\}_{n=0}^\infty$ is an orthogonal sequence, $\mathcal{R}((I - V^n V^{*n})^{1/2}) = \mathcal{N}(V^{*n})$ for all $n \in \mathbb{N}$ and

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Example

One can show that $e^{-\|b\|^2/2}\, C_\varphi$ is a coisometry. In particular, C_φ is cohyponormal. Hence, C_φ is normaloid and consequently, by our main theorem, we have

$$r(C_\varphi) = \|C_\varphi\| = e^{\|b\|^2/2}. \tag{3}$$

Note that C_φ is not normal (because if C_φ is hyponormal, then $b = \varphi(0) = 0$).

- In other words, if $\dim \mathcal{H} \geqslant \aleph_0$, then there always exists bounded non-normal cohyponormal composition operators in $\exp(\mathcal{H})$.

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