Krein space representation of submodules in $H^2(\mathbb{D}^2)$

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Introduction

$\mathbb{D}$: the open unit disk in $\mathbb{C}$,

$\mathbb{T}$: the boundary of $\mathbb{D}$.

My interest

I have been interested in $(\varphi_1, \varphi_2, \varphi_3)$ satisfying

1. $\varphi_1, \varphi_2, \varphi_3$ are holomorphic on $\mathbb{D}^2$,
2. $|\varphi_1(\lambda)|^2 + |\varphi_2(\lambda)|^2 - |\varphi_3(\lambda)|^2 \leq 1 \quad (\lambda \in \mathbb{D}^2)$,
3. $|\varphi_1(\lambda)|^2 + |\varphi_2(\lambda)|^2 - |\varphi_3(\lambda)|^2 \to 1$ a.e. as λ tends radially to $\mathbb{T}^2$.

Introduction (in $\mathbb{D}$)

Inner function

φ is called an inner function if

1. φ is holomorphic on $\mathbb{D}$,
2. $|\varphi(\lambda)|^2 \leq 1 \quad (\lambda \in \mathbb{D})$,
3. $|\varphi(\lambda)|^2 \to 1$ a.e. as λ tends radially to $\mathbb{T}$.

The following functions are inner:

- $(z - \lambda)/(1 - \overline{\lambda} z) \quad (\lambda \in \mathbb{D})$,
- $\exp((z + e^{i\theta})/(z - e^{i\theta})) \quad (\theta \in [0, 2\pi))$.

Introduction (my interest again)

My interest

1. $\varphi_1, \varphi_2, \varphi_3$ are holomorphic on $\mathbb{D}^2$,
2. $|\varphi_1(\lambda)|^2 + |\varphi_2(\lambda)|^2 - |\varphi_3(\lambda)|^2 \leq 1 \quad (\lambda \in \mathbb{D}^2)$,
3. $|\varphi_1(\lambda)|^2 + |\varphi_2(\lambda)|^2 - |\varphi_3(\lambda)|^2 \to 1$ a.e. as λ tends radially to $\mathbb{T}^2$.

Examples

Trivial example

$$(\varphi_1, \varphi_2, \varphi_3) = (z_1, z_2, z_1 z_2)$$

$\because$ For $\lambda_1, \lambda_2 \in \mathbb{D}$,

$$1 - (|\lambda_1|^2 + |\lambda_2|^2 - |\lambda_1 \lambda_2|^2) = (1 - |\lambda_1|^2)(1 - |\lambda_2|^2) \geq 0.$$

and

$$|\lambda_1|^2 + |\lambda_2|^2 - |\lambda_1 \lambda_2|^2 \to 1 + 1 - 1 = 1.$$

Examples

Non-trivial example

For inner functions $q_0(z_1)$ and $q_1(z_1)$ on $\mathbb{D}$ satisfying

1. q_0/q_1 is also inner,
2. $\alpha := \sqrt{1 - |(q_0/q_1)(0)|^2} \neq 0$.

$$(\varphi_1, \varphi_2, \varphi_3) := (q_0, \frac{-\sqrt{1-\alpha}q_0 + \sqrt{1+\alpha}q_1}{\sqrt{2\alpha}} z_2, \frac{\sqrt{1+\alpha}q_0 - \sqrt{1-\alpha}q_1}{\sqrt{2\alpha}} z_2)$$

How to find examples

$H^2(\mathbb{D}^2)$: the Hardy space over $\mathbb{D}^2$,

$$H^2(\mathbb{D}^2) \text{ is a Hilbert module over } \mathbb{C}[z_1, z_2].$$

$$\downarrow$$

$$\mathcal{M} \subset H^2 \text{ (a submodule)}$$

$$\downarrow$$

$$\Delta_{\mathcal{M}} \text{ (the defect operator of } \mathcal{M})$$

$$\downarrow$$

$$\text{if rank}\, \Delta_{\mathcal{M}} = 3$$

$$\downarrow$$

$$\Delta_{\mathcal{M}} = \varphi_1 \otimes \varphi_1 + \varphi_2 \otimes \varphi_2 - \varphi_3 \otimes \varphi_3$$

(the spectral resolution of $\Delta_{\mathcal{M}}$)

Further examples

In general, $\operatorname{rank} \Delta_{\mathcal{M}} = 2N+1$ $(N = 0, 1, 2, \ldots, \infty)$ $(R. Yang)$.

Hence we have the following cases:

- $|\varphi_1(\lambda)|^2 \le 1$
- $|\varphi_1(\lambda)|^2 + |\varphi_2(\lambda)|^2 - |\varphi_3(\lambda)|^2 \le 1$
- $|\varphi_1(\lambda)|^2 + |\varphi_2(\lambda)|^2 + |\varphi_3(\lambda)|^2 - |\varphi_4(\lambda)|^2 - |\varphi_5(\lambda)|^2 \le 1$ (← hard)
- $\vdots$

Setting

In our construction, $(\varphi_1, \varphi_2, \varphi_3)$ has the following additional properties:

Additional properties

- $\varphi_j \in H^2(\mathbb{D}^2)$,
- φ_i and φ_j are orthogonal in $H^2(\mathbb{D}^2)$.

Remark

- φ_j might be unbounded (by Rudin).

Setting

- we deal with $(\varphi_1, \varphi_2, \varphi_3)$ obtained by our construction,
- we will assume that each φ_j is bounded.

Theorem 1 (representation of modules)

Let $\mathcal{M}$ be a submodule in $H^2(\mathbb{D}^2)$ with $(\varphi_1, \varphi_2, \varphi_3)$. Then

$\exists \mathcal{K}$ ($= \mathcal{H}_+ \oplus \mathcal{H}_-$): a Krein space s.t. $\dim \mathcal{K} = 3$

$\exists D : \mathcal{K} \otimes H^2(\mathbb{D}^2) \to \mathcal{M}$ a module map s.t.

$$DD^\sharp = P_{\mathcal{M}} = T_{\varphi_1} T_{\varphi_1}^* + T_{\varphi_2} T_{\varphi_2}^* - T_{\varphi_3} T_{\varphi_3}^*.$$

Further,

$$I_{H^2} = T_{\varphi_1}^* T_{\varphi_1} + T_{\varphi_2}^* T_{\varphi_2} - T_{\varphi_3}^* T_{\varphi_3}.$$

Remark (Beurling)

In $H^2(\mathbb{D})$, $\mathcal{M}$ is a submodule iff $\mathcal{M} = \varphi H^2(\mathbb{D})$ where φ is inner.

Further, $P_{\mathcal{M}} = T_\varphi T_\varphi^*$ and $I_{H^2} = T_\varphi^* T_\varphi$.

Theorem 2 (homomorphisms)

$U = (u_{ij})$: a J-inner matrix valued function (that is, $UJU^* = J$ a.e. on $\mathbb{T}^2$ and $UJU^* \leq J$ on $\mathbb{D}^2$) with H^∞-entries.

Then $\exists \mathbb{U}$: a module map on $\mathcal{K} \otimes H^2$ s.t.

1. $\mathbb{U}^\sharp \mathbb{U} = I_{\mathcal{K} \otimes H^2}$, that is, $\mathbb{U}$ is $\sharp$-isometric,
2. $\|(\mathbb{U}F)(\lambda)\|_{\mathcal{K}}^2 \leq \|F(\lambda)\|_{\mathcal{K}}^2$ for any λ in $\mathbb{D}^2$.

The converse is also true if $\mathbb{U}$ is continuous.

Back to examples

If

$$(\varphi_1, \varphi_2, \varphi_3) := (q_0, \frac{-\sqrt{1-\alpha}q_0 + \sqrt{1+\alpha}q_1}{\sqrt{2\alpha}} z_2, \frac{\sqrt{1+\alpha}q_0 - \sqrt{1-\alpha}q_1}{\sqrt{2\alpha}} z_2)$$

then

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{1+\alpha}}{\sqrt{2\alpha}} & -\frac{\sqrt{1-\alpha}}{\sqrt{2\alpha}} \\ 0 & -\frac{\sqrt{1-\alpha}}{\sqrt{2\alpha}} & \frac{\sqrt{1+\alpha}}{\sqrt{2\alpha}} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 z_2 \\ q_0 z_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}.$$

Theorem 3 (local representation)

For any $(\varphi_1, \varphi_2, \varphi_3)$,

$\exists\mathcal{H}$: a Hilbert space,

$\exists V = (V_{ij}) : \mathbb{C} \oplus \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H} \to \mathbb{C} \oplus \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}$, an isometry

s.t. for $k = 1, 2$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2$,

$$\begin{aligned}\varphi_k(\lambda) = {} & V_{k1} + V_{k2}\varphi_3(\lambda) \\ & + (\lambda_1 V_{k3} + \lambda_2 V_{k4})(I_{\mathcal{H}} - \lambda_1 V_{33} - \lambda_2 V_{34})^{-1}(V_{31} + V_{32}\varphi_3(\lambda))\end{aligned}$$

where $|\lambda_1|$ and $|\lambda_2|$ are sufficiently small.

Summary

- In operator theory on $H^2(\mathbb{D}^2)$, Krein space approach will be useful as Yang suggested[2] to me.
- Triplet $(\varphi_1, \varphi_2, \varphi_3)$ will be manageable. However, we should not avoid unbounded cases toward general theory.
- Our approach can be applied to other spaces.

[2] His approach is different from that given here.