Curvature Inequalities for Operators in the Cowen-Douglas Class of a Planar Domain

Md. Ramiz Reza

Department of Mathematics
Indian Institute of Science
Bangalore

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Let Ω be a bounded domain in $\mathbb{C}$. Assume that $\partial\Omega$ consists of $n+1$ analytic Jordan curves.
Set $\Omega^* = \{\bar{z} \mid z \in \Omega\}$. Here we study operators in $B_1(\Omega^*)$, first introduced by Cowen and Douglas.

Definition 1 (Cowen-Douglas class of operators: (1978))

An operator T acting on a complex separable Hilbert space $\mathcal{H}$, is said to be in the class $B_1(\Omega^*)$ if it meets the following requirements:

1. $\mathrm{ran}(T - w) = \mathcal{H}, \quad w \in \Omega^*$,
2. $\bigvee_{w\in\Omega^*} \ker(T - w) = \mathcal{H}$ and
3. $\dim(\ker(T - w)) = 1, \quad w \in \Omega^*$.

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These conditions ensure the existence of a rank 1 Hermitian holomorphic vector bundle E_T over Ω^*, that is,

$$E_T := \{(w, v) \in \Omega^* \times \mathcal{H} : v \in \mathrm{Ker}(T - w)\}, \pi(w, v) = w$$

and there exist a holomorphic frame $w \to \gamma(w)$ with the property $\mathrm{Ker}(T - w) = \mathrm{span}\ \{\gamma(w)\}$.

- The Hermitian structure of the vector bundle E_T at the point w with respect to the frame $\gamma(w)$ is obtained from that of the subspace $\ker(T-w)$ of the Hilbert space $\mathcal{H}$ and we denote it by $h(w)=\langle\gamma(w),\gamma(w)\rangle_{\mathcal{H}}$.

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The curvature K_γ of the bundle E_T w.r.t the frame γ is given by the following formula

$$\begin{aligned}\mathrm{K}_\gamma(w) &= \frac{\partial}{\partial \bar{w}}\big(h^{-1}(w)\frac{\partial}{\partial w}h(w)\big)d\bar{w}\wedge dw\\ &= \frac{\partial}{\partial w}\frac{\partial}{\partial \bar{w}}\log(h(w))d\bar{w}\wedge dw = \mathcal{K}_\gamma(w)dw\wedge d\bar{w}\end{aligned}$$

$$\mathcal{K}_\gamma(w) = -\frac{\partial}{\partial w}\frac{\partial}{\partial \bar{w}}\log(h(w)) = -\frac{\|\gamma(w)\|^2\|\gamma\prime(w)\|^2 - |\langle\gamma(w),\gamma\prime(w)\rangle|^2}{\|\gamma(w)\|^4}.$$

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- The expression of the curvature $\mathcal{K}_\gamma(w)$ is independent of the choice of the frame γ. So, we call it curvature of the operator T and denote it by $\mathcal{K}_T(w)$.

Theorem 2 (Cowen-Douglas)

Two operators T_1 and T_2 in $B_1(\Omega^*)$ are unitarily equivalent if and only if the associated bundle E_{T_1} and E_{T_2} are equivalent as Hermitian holomorphic vector bundle if and only if $\mathcal{K}_{T_1}(w) = \mathcal{K}_{T_2}(w)$ for every w in Ω^*.

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- Every $T \in B_1(\Omega^*)$ is unitarily equivalent to the adjoint M^* of the operator of multiplication M by the coordinate function on some Hilbert space $\mathcal{H}_K$ consisting of holomorphic function on Ω possessing a reproducing kernel K.
- The kernel K is complex valued function defined on $\Omega \times \Omega$, which is holomorphic in the first and anti-holomorphic in the second variable and is positive semi-definite in the sense that $\left(\!\left(K(z_i, z_j) \right)\!\right)$ is positive semi-definite for every subset $\{z_1, \ldots, z_n\}$ of Ω.

The curvature $\mathcal{K}_T$ of the operator T is then equal to

$$\begin{aligned}\mathcal{K}_T(\bar{w}) &= -\frac{\partial^2}{\partial z \partial \bar{z}} \log K_T(z,z)|_{z=w} \\ &= -\frac{\|K_w\|^2 \left\|\bar{\partial} K_w\right\|^2 - |\langle K_w, \bar{\partial} K_w\rangle|^2}{(K(w,w))^2}, \quad w \in \Omega.\end{aligned}$$

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- The space $\ker(M^* - \bar{w})^2 = \operatorname{span}\{K_w, \bar{\partial} K_w\}$ is an invariant subspace for M^*.
- Representing the restriction of the operator M^* to this subspace with respect to an orthonormal basis as a 2×2 matrix, we have

$$M^*_{|\ker(M^* - \bar{w})^2} = \begin{pmatrix} \bar{w} & \frac{1}{\sqrt{-\mathcal{K}_T(\bar{w})}} \\ 0 & \bar{w} \end{pmatrix}.$$

Spectral set

Definition 3 (Spectral Set)

A compact subset $X \subseteq \mathbb{C}$ is said to be a spectral set for an operator A in $\mathcal{L}(\mathcal{H})$, if $\sigma(A) \subseteq X$ and the homomorphism $\rho_A : \mathrm{Rat}(X) \to \mathcal{L}(\mathcal{H})$ defined by $\rho_A(r) = r(A)$ is contractive.

where $\mathrm{Rat}(X)$ denotes the algebra of rational functions whose poles are off X, equipped with sup norm on X.

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Now assume $\overline{\Omega^*}$, the closure of Ω^*, is a spectral set for an operator M^* in $B_1(\Omega^*)$. For an arbitrary fixed point $w \in \Omega$ and $r \in \mathrm{Rat}(\overline{\Omega^*})$, we have

$$r(M^*)_{|\ker(M^*-\bar{w})^2} = \begin{pmatrix} r(\bar{w}) & \frac{r'(\bar{w})}{\sqrt{-\mathcal{K}_T(\bar{w})}} \\ 0 & r(\bar{w}) \end{pmatrix}.$$

$$\sup\{\frac{|r'(\bar{w})|}{1-|r(\bar{w})|^2} : \|r\|_\infty \leq 1,\ r \in \mathrm{Rat}(\overline{\Omega^*})\} = 2\pi(S_{\Omega^*}(\bar{w},\bar{w})),$$

where $S_{\Omega^*}(z,w)$, the Sżego kernel of Ω^*, is the reproducing kernel for the Hardy space $(H^2(\Omega^*), ds)$.

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- Contractivity condition gives a curvature inequality :

Curvature Inequality 1

$\mathcal{K}_T(\bar{w}) \leq -4\pi^2(S_{\Omega^*}(\bar{w},\bar{w}))^2, \quad \bar{w} \in \Omega^*.$

Equivalently, since $S_\Omega(z,w) = S_{\Omega^*}(\bar{w},\bar{z})$, the curvature inequality takes the form

Curvature Inequality 2

$\frac{\partial^2}{\partial w \partial \bar{w}} log K_T(w,w) \geq 4\pi^2(S_\Omega(w,w))^2, \quad w \in \Omega.$

- The reproducing kernel of the Hardy space, as is well-known, is the Sz̈ego kernel $S_{\mathbb{D}}$ of the unit disc $\mathbb{D}$. It is given by the formula $S_{\mathbb{D}}(z, w) = \frac{1}{2\pi(1-z\bar{w})}$, for all z, w in $\mathbb{D}$.

$$\mathcal{K}_{M^*}(w) = -\frac{\partial^2}{\partial w \partial \bar{w}} \log S_{\mathbb{D}}(w, w) = -4\pi^2 (S_{\mathbb{D}}(w, w))^2, \quad w \in \mathbb{D}.$$

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- If the region Ω is simply connected, then using the Riemann map and the transformation rule for the Sz̈ego kernel together with the chain rule for composition, we see that

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log S_{\Omega}(w, w) = 4\pi^2 (S_{\Omega}(w, w))^2, \quad w \in \Omega.$$

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Pointwise equality and a uniqueness question of Douglas

For a contraction T in $B_1(\mathbb{D})$, if $\mathcal{K}_T(\zeta) = -4\pi^2 S_{\mathbb{D}}(\zeta,\zeta)^2$ for some fixed ζ in $\mathbb{D}$, then does it follow that T must be unitarily equivalent to M^* on $(H^2(\mathbb{D}), ds)$?

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- Answer: No.
- Misra('84) provided a counterexample.

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However, he proved that

Theorem 4

For a homogeneous contraction T in $B_1(\mathbb{D})$, if $\mathcal{K}_T(\zeta) = -4\pi^2 S_{\mathbb{D}}(\zeta,\zeta)^2$ for some fixed ζ in $\mathbb{D}$, then T must be unitarily equivalent to M^* on $(H^2(\mathbb{D}), ds)$.

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Theorem 5

Let T be an operator in $B_1(\mathbb{D})$ whose adjoint is a subnormal contraction. Let ζ be a fixed but arbitrary point in $\mathbb{D}$. Assume that polynomials are dense in $\mathcal{H}_{\phi_\zeta(T)}$ and that $\mathcal{K}_T(\zeta) = -\frac{1}{(1-|\zeta|^2)^2}$, then T is unitarily equivalent to U_+^*, the standard unilateral backward shift operator. (where $\mathcal{H}_{\phi_\zeta(T)}$ is the normalized reproducing kernel associated to the operator $\phi_\zeta(T)$)

On the other hand, if the region is not simply connected, then Suita('76) has shown that

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log S_\Omega(w,w) > 4\pi^2 (S_\Omega(w,w))^2, \quad w \in \Omega.$$

- If Ω is not simply connected, then the operator M^* on $(H^2(\Omega), ds)$ fails to be extremal.
- We don't know if there exists an operator T in $B_1(\Omega^*)$, admitting $\overline{\Omega^*}$ as a spectral set for which

$$\mathcal{K}_T(w) = -4\pi^2 (S_{\Omega^*}(w,w))^2, \quad w \in \Omega^*.$$

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Uniqueness in finitely connected domain

Fix a point ζ in Ω^*. We have already seen the existence of a co-subnormal operator M_ζ^* such that $\mathcal{K}_{M_\zeta^*}(\zeta) = -4\pi^2(S_{\Omega^*}(\zeta,\zeta))^2$.

- In fact M_ζ is a pure, rationally cyclic, subnormal operator whose spectrum equal to $\overline{\Omega}$ and normal spectrum equal to $\partial\Omega$.

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- In fact M_ζ is a pure, rationally cyclic, subnormal operator whose spectrum equal to $\overline{\Omega}$ and normal spectrum equal to $\partial\Omega$.

Abrahamse and Douglas('76) has characterized the unitary equivalence class of such operator. These are called bundle shift of multiplicity one. Their results also proves that adjoint of each of the bundle shift lies in $B_1(\Omega^*)$ having $\overline{\Omega^*}$ as a spectral set.

Bundle shift of multiplicity 1:

Let α be an element in $\mathrm{Hom}(\pi_1(\Omega), \mathbb{T})$. Such a homomorphism is also called a character. Each of these character induces a flat unitary bundle E_α of rank 1 on Ω.

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Two rank one flat unitary vector bundle E_α and E_β are equivalent as a flat unitary vector bundle if and only if their inducing characters are equal that is $\alpha = \beta$.

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Two rank one flat unitary vector bundle E_α and E_β are equivalent as a flat unitary vector bundle if and only if their inducing characters are equal that is $\alpha = \beta$.

Fix a character α. Let $H^2_{E_\alpha}$ be the linear space of those holomorphic sections f of E_α such that the subharmonic function $\|f(z)\|^2$ on Ω is majorized by a harmonic function on Ω.

Fix a point $p \in \Omega$. Then the norm of a section f in $H^2_\alpha(\Omega)$ is defined by

$$\|f\|^2 = \int_{\partial\Omega} \|f(z)\|^2 \, ds.$$

- The linear space $(H^2_{E_\alpha}, ds)$ is complete with respect to this norm making it into a Hilbert space.
- A bundle shift T_{E_α} is simply the operator of multiplication by the coordinate function on $(H^2_{E_\alpha}, ds)$.

Theorem 7 (Abrahamse and Douglas'76)

Let E_α and E_β be two rank one flat unitary vector bundles induced by the homomorphisms α and β respectively. Then the bundle shift T_{E_α} is unitarily equivalent to the bundle shift T_{E_β} if and only if E_α and E_β are equivalent as flat unitary vector bundles.

- It is not very hard to verify that T_{E_α} is a pure cyclic subnormal operator with spectrum $\overline{\Omega}$ and normal spectrum $\partial\Omega$.

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- It is not very hard to verify that T_{E_α} is a pure cyclic subnormal operator with spectrum $\overline{\Omega}$ and normal spectrum $\partial\Omega$.

Theorem 8 (Abrahamse and Douglas '76)

Every pure cyclic subnormal operator with spectrum $\overline{\Omega}$ and normal spectrum contained in $\partial\Omega$ is unitarily equivalent to a bundle shift T_{E_α} for some character α.

Weighted Hardy space and bundle shift

Let λ be a positive continuous function on $\partial\Omega$. Define an equivalent norm on $H^2(\Omega)$ in the following way

$$\|f\|^2_{\lambda ds} = \int_{\partial\Omega} |f(z)|^2 \lambda(z) ds(z).$$

Let $\left(H^2(\Omega), \lambda ds\right)$ denote the linear space $H^2(\Omega)$ endowed with the norm λds.

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- the operator M on it is cyclic, pure subnormal, its spectrum is equal to $\overline{\Omega}$ and finally its normal spectrum is equal to $\partial\Omega$.

Weighted Hardy space and bundle shift

- The operator M on $\left(H^2(\Omega), \lambda ds\right)$ must be unitarily equivalent to the bundle shift T_{E_α} on $\left(H^2_{E_\alpha}(\Omega), ds\right)$ for some character α.

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The character α is determined by the following n-tuple of numbers:

$$c_j(\lambda) = -\int_{\partial\Omega_j} \frac{\partial}{\partial\eta_z}\big(u_\lambda(z)\big)\, ds(z), \quad \text{for } j = 1, 2, \ldots, n,$$

where u_λ is the harmonic function on Ω with continuous boundary value $\frac{1}{2}\log\lambda$.

$$\alpha = \big(\exp\big(ic_1(\lambda)\big), \ldots, \exp\big(ic_n(\lambda)\big)\big)$$

Weighted Hardy space and bundle shift

Theorem 9

Let λ, μ be two positive continuous function on $\partial\Omega$. Then the operators M on the Hilbert spaces $\big(H^2(\Omega), \lambda ds\big)$ and $\big(H^2(\Omega), \mu ds\big)$ are unitarily equivalent if and only if

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- Following a result of Abrahamse, it follows that, given any character α, there exists a positive continuous function λ defined on $\partial\Omega$ such that the operator M on $\left(H^2(\Omega), \lambda ds\right)$ is unitarily equivalent to the bundle shift M on $\left(H^2_{E_\alpha}(\Omega), ds\right)$.

Weighted kernel and extremal operator at a point

- $\big(H^2(\Omega), \lambda ds\big)$ is a reproducing kernel Hilbert space.

Let $K^{(\lambda)}(z,w)$ denote the kernel function for $\big(H^2(\Omega), \lambda ds\big)$.

Weighted kernel and extremal operator at a point

- $\left(H^2(\Omega), \lambda ds\right)$ is a reproducing kernel Hilbert space.

Let $K^{(\lambda)}(z, w)$ denote the kernel function for $\left(H^2(\Omega), \lambda ds\right)$. The case $\lambda \equiv 1$ gives us the Sżego kernel $S(z, w)$ for the domain Ω. Associated to the Sżego kernel, there exists a conjugate kernel $L(z, w)$, called the Garabedian kernel, which is related to the Sżego kernel via the following identity.

$$\overline{S(z, w)}ds = \frac{1}{i}L(z, w)dz, \quad w \in \Omega \text{ and } \ z \in \partial\Omega$$

- For each fixed w in Ω, the function $S_w(z)$ is holomorphic in a neighbourhood of Ω and $L_w(z)$ is holomorphic in a neighbourhood of $\Omega - \{w\}$ with a simple pole at w. $L_w(z)$ is non vanishing on $\bar{\Omega} - \{w\}$. The function $S_w(z)$ is non vanishing on $\partial\Omega$ and has exactly n zeros in Ω.

Weighted kernel and extremal operator at a point

Nehari has extended these result for the kernel $K^{(\lambda)}(z,w)$.

Theorem 10 (Nehari'50)

Let λ be a positive continuous function on $\partial\Omega$. Then there exist two analytic function $K^{(\lambda)}(z,w)$ and $L^{(\lambda)}(z,w)$ with the following properties: for each fixed w in Ω, the function $K_w^{(\lambda)}(z)$ and $L_w^{(\lambda)}(z) - (2\pi(z-w))^{-1}$ are holomorphic in Ω; $|K_w^{(\lambda)}(z)|$ is continuous on $\bar{\Omega}$ and $|L_w^{(\lambda)}(z)|$ is continuous in $\bar{\Omega} - C_\epsilon$, where C_ϵ denotes a small open disc about w; $K_w^{(\lambda)}(z)$ and $L_w^{(\lambda)}(z)$ are connected by the identity

$$\overline{K_w^{(\lambda)}(z)}\lambda(z)ds = \frac{1}{i}L_w^{(\lambda)}(z)dz, \quad w \in \Omega \text{ and } z \in \partial\Omega$$

These properties determine both functions uniquely.

Characterization of extremal operator

- Let $\{a_1, \dots, a_n\}$ be the zeros of the Sżego kernel $S_\zeta(z)$ in Ω.

Theorem 11

The operator M^* on the Hilbert space $\left(H^2(\Omega), \lambda ds\right)$ is extremal at $\bar{\zeta}$, that is, $\frac{\partial^2}{\partial w \partial \bar{w}} log K^{(\lambda)}(w, w)_{|w=\zeta} = 4\pi^2 (S_\Omega(\zeta, \zeta))^2$ if and only if $L_\zeta^{(\lambda)}(z)$ has $\{a_1, a_2, \dots, a_n\}$ as the zero set.

- Consider the closed convex set M_1 in $\big(H^2(\Omega), \lambda(z)ds\big)$ defined by

$$M_1 := \{f \in \big(H^2(\Omega), \lambda(z)ds\big) : f(\zeta) = 0, f'(\zeta) = 1\}.$$

- Now consider the extremal problem of finding

$$\inf\ \{\|f\|^2 : f \in M_1\}.$$

- The unique function F in $\big(H^2(\Omega), \lambda(z)ds\big)$ is a solution to the extremal problem iff $F \in M_1$ and F is orthogonal to the subspace

$$\begin{aligned} H_1 = \{f \in \big(H^2(\Omega), \lambda(z)ds\big) : f(\zeta) = 0, f'(\zeta) = 0\} \\ = \big(\mathrm{Span}\{K_\zeta^{(\lambda)}, \bar{\partial}K_\zeta^{(\lambda)}\}\big)^\perp. \end{aligned}$$

- $\inf\{\|f\|^2 : f \in M_1\} = \frac{1}{K^{(\lambda)}(\zeta,\zeta)}\left(\frac{\partial^2}{\partial w \partial \bar{w}} \log K^{(\lambda)}(w,w)|_{w=\zeta}\right)^{-1}.$
- Now consider the function g in M_1 defined by

$$g(z) := \frac{K_\zeta^{(\lambda)}(z) F_\zeta(z)}{2\pi S(\zeta,\zeta) K^{(\lambda)}(\zeta,\zeta)}, \quad z \in \Omega,$$

  where $F_\zeta(z) = \frac{S_\zeta(z)}{L_\zeta(z)}$ denote the Ahlfors map for the domain Ω at the point ζ
- Using the reproducing property for the kernel function $K^{(\lambda)}$ and the fact that $|F_\zeta(z)| \equiv 1$ on $\partial\Omega$, it is straightforward to verify that

$$\|g\|^2_{\lambda ds} = \left(K^{(\lambda)}(\zeta,\zeta) 4\pi^2 S(\zeta,\zeta)^2\right)^{-1}.$$

- Consequently we have,

$$\frac{\partial^2}{\partial w \partial \bar{w}} \log K^{(\lambda)}(w,w) \mid_{w=\zeta} \geq 4\pi^2 (S_\Omega(\zeta,\zeta))^2.$$

- So equality holds iff g solve the extremal problem iff g is orthogonal to the subspace H_1.
- g is orthogonal to H_1 iff I_f vanishes for all $f \in H_1$.
- $I_f = \int_{\partial\Omega} f(z)\overline{K_\zeta^{(\lambda)}(z)}\,\overline{F_\zeta(z)}\lambda(z)ds = \frac{1}{i}\int_{\partial\Omega} f(z)\overline{F_\zeta(z)} L_\zeta^{(\lambda)}(z)dz$
$= \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(z)L_\zeta^{(\lambda)}(z)\left(2\pi L_\zeta(z)\right)}{S_\zeta(z)}dz$
- Using residue theorem, we get that I_f vanishes for all $f \in H_1$ iff the set of zeros of the function $L_\zeta^{(\lambda)}(z)$ in Ω is $\{a_1, a_2, \ldots, a_n\}$.

Example of extremal operator at a point

$$\lambda(z) := \prod_{k=1}^{n} |z - a_k|^2, \quad z \in \partial\Omega.$$

Then, for $z \in \partial\Omega$, we have

$$\frac{\overline{S_\zeta(z)}}{\prod_{j=1}^{n}(\bar{z} - \bar{a}_j)(\zeta - a_j)}\lambda(z)ds = \frac{1}{i}\frac{\prod_{k=1}^{n}(z - a_k)}{\prod_{k=1}^{n}(\zeta - a_k)}L_\zeta(z)dz$$

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Hence using the uniqueness part of the Nehari's Theorem, we get

$$K_\zeta^{(\lambda)}(z) = \frac{S_\zeta(z)}{\prod_{j=1}^{n}(z - a_j)(\bar{\zeta} - \bar{a}_j)}, L_\zeta^{(\lambda)}(z) = \frac{\prod_{k=1}^{n}(z - a_k)}{\prod_{k=1}^{n}(\zeta - a_k)}L_\zeta(z),$$

Clearly, $\{a_1, a_2, \ldots, a_n\}$ is the zero set of the function $L_\zeta^{(\lambda)}(z)$.

Another example of extremal operator at a point

$$\lambda(z)ds = \frac{|S_\zeta(z)|^2}{S(\zeta,\zeta)}ds, \ \ z \in \partial\Omega,$$

Using the reproducing property of the Sżego kernel, it is easy to verify that

$$\left\langle f, 1 \right\rangle_{\left(H^2(\Omega),\lambda ds\right)} = f(\zeta)$$

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This gives us $K_\zeta^{(\lambda)}(z) \equiv 1$. Now for $z \in \partial\Omega$, we have,

$$\lambda(z)ds = \frac{S_\zeta(z)}{S(\zeta,\zeta)}\overline{S_\zeta(z)}ds = \frac{1}{i}\frac{S_\zeta(z)}{S(\zeta,\zeta)}L_\zeta(z)dz,$$

Again, using the uniqueness in Nehari's Theorem we get

$$L_\zeta^{(\lambda)}(z) = S_\zeta(z)L_\zeta(z)\big(S(\zeta,\zeta)\big)^{-1}, \qquad z \in \bar{\Omega} - \{\zeta\}.$$

Again, the zero set of the function $L_\zeta^{(\lambda)}(z)$ is $\{a_1, a_2, \dots, a_n\}$.

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- In both cases, the character is determined by the following n-tuple of number

$$\left\{ \exp\left(2\pi i(1-\omega_1(\zeta))\right), \dots, \exp\left(2\pi i(1-\omega_n(\zeta))\right) \right\}.$$

where $\omega_i(z)$ is the harmonic function on Ω with boundary value equal to 1 when $z \in \partial\Omega_i$ and equal to 0, when $z \in \partial\Omega/\partial\Omega_i$.

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- In fact we have proved the following uniqueness theorem.

Uniqueness of extremal operator

Theorem 12 (uniqueness of extremal bundle shift)

If the operator M^* on the Hilbert space $\big(H^2(\Omega), \lambda ds\big)$ is extremal at $\bar{\zeta}$, then the operator M is unitarily equivalent to the bundle shift T_{E_α} on $\big(H^2_{E_\alpha}(\Omega), ds\big)$, where α is uniquely determined by the following n-tuple of complex number of unit modulus:

$$\Big\{ \exp\big(2\pi i(1-\omega_1(\zeta))\big), \ldots, \exp\big(2\pi i(1-\omega_n(\zeta))\big) \Big\}.$$

$$\sum_{j=1}^{n+1} \omega_j \equiv 1 \quad \text{and} \quad 0 < \omega_{n+1}(z) < 1, \;\; z \in \Omega.$$

- Therefore we get that $(n-1) < \sum_{j=1}^{n} \big(1-\omega_j(\zeta)\big) < n$.

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- Therefore we get that $(n-1) < \sum_{j=1}^{n} \big(1-\omega_j(\zeta)\big) < n$.
- For $n \geq 2$, the set of extremal operators does not include the adjoint of many of the bundle shifts.

Special case of doubly connected domain

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- In fact this is true of any doubly connected bounded domain Ω with Jordan analytic boundary. Since using connectedness argument, in such cases, again we will have $\omega_1(\Omega) = (0, 1)$.

Thank You