Sidon sets in duals of compact groups and generalizations

Gilles Pisier
Texas A&M University

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Sidon sets in the last century

$\Lambda \subset \mathbb{Z}$ is Sidon if

$$\sum_{n\in\Lambda} a_n e^{int} \in C(\mathbf{T}) \Rightarrow \sum_{n\in\Lambda} |a_n| < \infty$$

Sidon sets (and more generally **"thin** sets" e.g. **Helson** sets) were a very active subject in the 1960's and 1970's: Kahane, Varopoulos, Yves Meyer, Bonami +others (in France), Edwards & Gaudry (Australia), Figa-Talamanca (Italy), Rudin, Hewitt & Ross, Rider (USA), Hartman & Ryll-Nardzewski, Bożejko (Poland), Katznelson(Israel), Herz, Drury (Canada)...
The first period culminated with Sam Drury's solution of **"the Union problem"**:
The union of two Sidon sets is again a Sidon set.

3 definitions

$\Lambda \subset \mathbb{Z}$ is Sidon if

$$\sum_{n\in\Lambda} a_n e^{int} \in C(\mathbf{T}) \Rightarrow \sum_{n\in\Lambda} |a_n| < \infty$$

$\Lambda \subset \mathbb{Z}$ is randomly Sidon if

$$\sum_{n\in\Lambda} \pm a_n e^{int} \in C(\mathbf{T}) \ a.s. \Rightarrow \sum_{n\in\Lambda} |a_n| < \infty$$

$\Lambda \subset \mathbb{Z}$ is subGaussian if

$$\sum_{n\in\Lambda} |a_n|^2 < \infty \Rightarrow \int \exp |\sum_{n\in\Lambda} a_n e^{int}|^2 < \infty.$$

Sidon

$\Lambda \subset \mathbb{Z}$ is Sidon if

$$\sum_{n\in\Lambda} a_n e^{int} \in C(\mathbf{T}) \Rightarrow \sum_{n\in\Lambda} |a_n| < \infty$$

Equivalently: $\exists C$ such that $\forall A \subset \Lambda$ with $|A| < \infty$

$$\sum_{n\in A} |a_n| \leq C \|\sum_{n\in A} a_n e^{int}\|_\infty$$

Randomly Sidon

$\Lambda \subset \mathbb{Z}$ is randomly Sidon if

$$\sum_{n \in \Lambda} \pm a_n e^{int} \in C(\mathbf{T}) \ a.s. \Rightarrow \sum_{n \in \Lambda} |a_n| < \infty$$

Equivalently: $\exists C$ such that $\forall A \subset \Lambda$ with $|A| < \infty$

$$\sum\nolimits_{n \in A} |a_n| \leq C \text{Average}_{\pm} \| \sum\nolimits_{n \in A} \pm a_n e^{int} \|_{\infty}$$

subGaussian

$\Lambda \subset \mathbb{Z}$ is subGaussian if

$$\sum_{n\in\Lambda} |a_n|^2 < \infty \Rightarrow \int \exp|\sum_{n\in\Lambda} a_n e^{int}|^2 < \infty.$$

Equivalently: $\exists C$ such that

$$\|\sum_{n\in\Lambda} a_n e^{int}\|_{\psi_2} \leq C(\sum_{n\in\Lambda} |a_n|^2)^{1/2}$$

where

$$\psi_2(x) = \exp x^2 - 1$$

and $\|f\|_{\psi_2}$ is the norm in associated Orlicz space

In terms of $\Lambda(p)$-set

subGaussian $\Leftrightarrow \exists C\ \forall 2 \leq p < \infty$

$$\|\sum_{n\in\Lambda} a_n e^{int}\|_p \leq C\sqrt{p}(\sum_{n\in\Lambda} |a_n|^2)1/2$$

$\Leftrightarrow$ $\Lambda(p)$-set with constant $O(\sqrt{p})$ when $p \to \infty$

They are all equivalent !

Obviously Sidon $\Rightarrow$ randomly Sidon

Rudin (1961): Sidon $\Rightarrow$ subGaussian

Rider (1975) : Sidon $\Leftrightarrow$ randomly Sidon

P (1978) : Sidon $\Leftrightarrow$ subGaussian

Results hold more generally for any subset $\Lambda \subset \widehat{G}$ when G is any compact Abelian group

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(Note: This refines Drury's celebrated 1970 union Theorem)

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when G is any compact Abelian group

Examples

Hadamard lacunary sequences $n_1 < n_2 < \cdots < n_k, \cdots$ such that

$$\inf_k \frac{n_{k+1}}{n_k} > 1$$

Explicit example

$$n_k = 2^k$$

Basic Example: Quasi-independent sets
Λ is quasi-independent if all the sums

$$\{\sum_{n \in A} n \mid A \subset \Lambda, |A| < \infty\} \text{ are distinct numbers}$$

$$\text{quasi-independent} \Rightarrow \text{Sidon}$$

Main Open Problem
Is every **Sidon** set a finite union of **quasi-independent** sets ?

The recent rebirth

Bourgain and Lewko (arxiv 2015) wondered whether a group environment is needed for all the preceding

Question

What remains valid if $\Lambda \subset \widehat{G}$ is replaced by a *uniformly bounded* orthonormal system ?

Let $\Lambda = \{\varphi_n\} \subset L_\infty(T,m)$ orthonormal in $L_2(T,m)$ ((T,m) any probability space)

(i) We say that (φ_n) is Sidon with constant C if for any n and any complex sequence (a_k) we have

$$\sum_1^n |a_k| \le C\|\sum_1^n a_k\varphi_k\|_\infty.$$

(ii) We say that (φ_n) is randomly Sidon with constant C if for any n and any complex sequence (a_k) we have

$$\sum_1^n |a_k| \le C\text{Average}_{\pm 1}\|\sum_1^n \pm a_k\varphi_k\|_\infty,$$

(iii) Let $k \ge 1$. We say that (φ_n) is $\otimes^k$-Sidon with constant C if the system $\{\varphi_n(t_1)\cdots\varphi_n(t_k)\}$ (or equivalently $\{\varphi_n^{\otimes k}\}$) is Sidon with constant C in $L_\infty(T^k, m^{\otimes k})$.

Now assume merely that $\{\varphi_n\} \subset L_2(T,m)$.

(iv) We say that (φ_n) is subGaussian with constant C (or C-subGaussian) if for any n and any complex sequence (a_k) we have

$$\|\sum_1^n a_k\varphi_k\|_{\psi_2} \le C(\sum |a_k|^2)^{1/2}.$$

Here

$$\psi_2(x) = \exp x^2 - 1$$

and $\|f\|_{\psi_2}$ is the norm in associated Orlicz space
Note:

$$\|f\|_{\psi_2} \approx \sup_{2\le p<\infty} p^{-1/2}\|f\|_p$$

Again: We say that $\{\varphi_n\} \subset L_2(T,m)$ is subGaussian with constant C (or C-subGaussian) if for any n and any complex sequence (a_k) we have

$$\|\sum\nolimits_1^n a_k\varphi_k\|_{\psi_2} \le C(\sum |a_k|^2)^{1/2}.$$

Equivalently, assuming w.l.o.g. $\int \varphi_k = 0, \forall k$
$\exists C$ such that $\forall (a_k)$

$$\int \exp Re(\sum\nolimits_1^n a_k\varphi_k) \le \exp C^2 \sum |a_k|^2$$

Important remark: Standard i.i.d. (real or complex) Gaussian random variables are subGaussian (Fundamental example !)

Easy Observation : *Sidon* $\not\Rightarrow$ *subGaussian*

By a much more delicate example Bourgain and Lewko proved:

$$subGaussian \not\Rightarrow Sidon$$

However, they proved

Theorem

$$subGaussian \Rightarrow \otimes^5 - Sidon$$

Recall $\otimes^5 - Sidon$ *means*

$$\sum_1^n |a_k| \leq C \| \sum_1^n a_k \varphi_k(t_1) \cdots \varphi_k(t_5) \|_{L_\infty(T^5)}.$$

This generalizes my 1978 result that subGaussian implies Sidon
for characters ($\varphi_k(t_1) \cdots \varphi_k(t_5) = \varphi_k(t_1 \cdots t_5)$!)
They asked whether 5 can be replaced by 2 which would be optimal

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They asked whether 5 can be replaced by 2 which would be optimal

Indeed, it is so.

Theorem

For bounded orthonormal systems

$$subGaussian \Rightarrow \otimes^2 - Sidon$$

Recall $\otimes^2 - Sidon$ *means*

$$\sum\nolimits_1^n |a_k| \leq C\| \sum\nolimits_1^n a_k\varphi_k(t_1)\varphi_k(t_2)\|_{L_\infty(T^2)}.$$

Actually, we have more generally:

Theorem (1)

Let $(\psi_n^1), (\psi_n^2)$ be systems biorthogonal respectively to $(\varphi_n^1), (\varphi_n^2)$ on probability spaces $(T_1, m_1), (T_2, m_2)$ resp. and uniformly bounded respectively by C_1', C_2',
If $(\varphi_n^1), (\varphi_n^2)$ are subGaussian with constants C_1, C_2 then

$$\sum |a_n| \leq \alpha \ \mathrm{ess\,sup}_{(t_1,t_2)\in T_1\times T_2} \Big| \sum a_n \psi_n^1(t_1)\psi_n^2(t_2) \Big|,$$

where α is a constant depending only on C_1, C_2, C_1', C_2'.

To illustrate by a concrete (but trivial) example: take $\varphi_n^1 = \varphi_n^2 = g_n$ and $\psi_n^1 = \psi_n^2 = \mathrm{sign}(g_n)$

Ingredient 1:Talagrand's Theorem

The key new ingredient is a corollary of a powerful result due to **Michel Talagrand, Acta (1985)** (combined with a soft Hahn-Banach argument)

Let (g_n) be an i.i.d. sequence of standard (real or complex) Gaussian random variables

Theorem

Let (φ_n) be C-subGaussian in $L_1(m)$. Then $\exists T : L_1(\Omega, \mathbb{P}) \to L_1(m)$ with $\|T\| \leq KC$ (K a numerical constant) such that

$$\forall n \quad T(g_n) = \varphi_n$$

Projective and Injective Tensor Product norm

denoted respectively by $\| \cdot \|_\wedge$ and $\| \cdot \|_\vee$

Theorem

Let (φ_n^1) and (φ_n^2) $(1 \leq n \leq N)$ are subGaussian with constants C_1, C_2. Then for any $0 < \delta < 1$ there is a decomposition in $L_1(m_1) \otimes L_1(m_2)$ of the form

$$\sum\nolimits_1^N \varphi_n^1 \otimes \varphi_n^2 = t + r$$

satisfying

$$\|t\|_\wedge \leq w(\delta)$$

$$\|r\|_\vee \leq \delta,$$

where $w(\delta)$ depends only on δ and C_1, C_2.
Moreover

$$w(\delta) = O(\log((C_1 C_2)/\delta)$$

Bourgain and Lewko noticed that Slepian's classical comparison Lemma for Gaussian processes implies that randomly $\otimes^k$-Sidon and randomly Sidon are the same property, not implying Sidon. However, we could prove that this implies $\otimes^4$-Sidon:

Theorem (2)

Let (φ_n, ψ_n) be a biorthogonal system, with both sequences bounded in L_∞. The following are equivalent:

(i) *The system (ψ_n) is randomly Sidon.*

(ii) *The system (ψ_n) is $\otimes^4$-Sidon.*

(iii) *The system (ψ_n) is $\otimes^k$-Sidon for all $k \geq 4$.*

(iv) *The system (ψ_n) is $\otimes^k$-Sidon for some $k \geq 4$.*

This generalizes Rider's result that randomly Sidon implies Sidon for characters

Open question: What about $k = 2$ or $k = 3$?

The Return of Union Problem

Corollary (Union problem for unif.bded o.n. systems)

Let (φ_n, ψ_n) be a biorthogonal system, with both sequences bounded in L_∞. Assume that (ψ_n) is the union of two (or finitely many) Sidon systems. Then (ψ_n) is $\otimes^4$-Sidon.

End of Part I

References: Bourgain-Lewko's and all my papers are on arxiv.
Books:

1970: E. Hewitt and K. Ross, *Abstract harmonic analysis, Volume II, Structure and Analysis for Compact Groups, Analysis on Locally Compact Abelian Groups*, Springer, Heidelberg, 1970.
1975: J. López and K.A. Ross, *Sidon sets.* Lecture Notes in Pure and Applied Mathematics, Vol. 13. Marcel Dekker, Inc., New York, 1975.
1981: M.B. Marcus and G. Pisier, *Random Fourier series with Applications to Harmonic Analysis*, Annals of Math. Studies n°101, Princeton Univ. Press, 1981.
1985: J. P. Kahane, *Some random series of functions. Second edition*, Cambridge University Press, 1985.
2013: C. Graham and K. Hare, *Interpolation and Sidon sets for compact groups*. Springer, New York, 2013. xviii+249 pp.

Non-commutative case

G compact non-commutative group
$\widehat{G}$ the set of distinct irreps, $d_\pi = \dim(H_\pi)$
$\Lambda \subset \widehat{G}$ is called Sidon if $\exists C$ such that for any finitely supported family (a_π) with $a_\pi \in M_{d_\pi}$ $(\pi \in \Lambda)$ we have

$$\sum\nolimits_{\pi \in \Lambda} d_\pi \mathrm{tr}|a_\pi| \leq C \| \sum\nolimits_{\pi \in \Lambda} d_\pi \mathrm{tr}(\pi a_\pi) \|_\infty.$$

$\Lambda \subset \widehat{G}$ is called randomly Sidon if $\exists C$ such that for any finitely supported family (a_π) with $a_\pi \in M_{d_\pi}$ $(\pi \in \Lambda)$ we have

$$\sum\nolimits_{\pi \in \Lambda} d_\pi \mathrm{tr}|a_\pi| \leq C \mathbb{E} \| \sum\nolimits_{\pi \in \Lambda} d_\pi \mathrm{tr}(\varepsilon_\pi \pi a_\pi) \|_\infty$$

where (ε_π) are an independent family such that each ε_π is uniformly distributed over $O(d_\pi)$.
Important Remark (easy proof) Equivalent definitions:
• unitary matrices (u_π) uniformly distributed over $U(d_\pi)$
• Gaussian random matrices (g_π) normalized so that $\mathbb{E}\|g_\pi\| \approx 2$
$(\{d_\pi^{1/2} g_\pi \mid \pi \in \Lambda, 1 \leq i, j \leq d_\pi\}$ forms a standard Gaussian (real or complex) i.i.d. family

Fundamental example

$$G = \prod_{n\geq 1} U(d_n)$$

$$\Lambda = \{\pi_n \mid n \geq 1\}$$

$$\pi_n : \ G \to U(d_n) \quad n\text{-th coordinate}$$

$$C = 1 : \ \sum\nolimits_{n\geq 1} d_n \mathrm{tr}|a_n| = \|\sum\nolimits_{n\geq 1} d_n \mathrm{tr}(\pi_n a_n)\|_\infty .$$

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Observe that for the functions $\varphi_n(i,j)$ defined on (G, m_G) by

$$\varphi_n(i,j)(g) = \pi_n(g)_{ij}$$

$\{d_n^{1/2}\varphi_n(i,j) \mid n \geq 1, 1 \leq i,j \leq d_n\}$ is an orthonormal system.

Rider (1975) extended all results previously mentioned to arbitrary compact groups
Note however that the details of his proof that randomly Sidon implies Sidon (solving the non-commutative union problem) never appeared
I posted a paper on this on arxiv including (presumably) his proof

General matricial systems

Assume given a sequence of finite dimensions d_n.
For each n let (φ_n) be a random matrix of size $d_n \times d_n$ on (T, m).
We call this a "matricial system":

$$\varphi_n = [\varphi_n(i,j)]$$

or rather for $t \in T$

$$\varphi_n(t) = [\varphi_n(i,j)(t)]$$

The **subGaussian condition** becomes: for any N and $y_n \in M_{d_n}$ ($n \le N$) we have

$$\|\sum d_n \mathrm{tr}(y_n \varphi_n)\|_{\psi_2} \le C(\sum d_n \mathrm{tr}|y_n|^2)^{1/2} = \|\sum d_n \mathrm{tr}(y_n g_n)\|_2. \tag{1}$$

In other words, $\{d_n^{1/2}\varphi_n(i,j) \mid n \ge 1, 1 \le i,j \le d_n\}$ is a subGaussian system of functions.

The **uniform boundedness condition** becomes

$$\exists C' \ \forall n \quad \|\varphi_n\|_{L_\infty(M_{d_n})} \le C'. \tag{2}$$

As for the **orthonormality condition** it becomes

$$\int \varphi_n(i,j)\overline{\varphi_{n'}(k,\ell)} = d_n^{-1}\delta_{n,n'}\delta_{i,k}\delta_{j,\ell}. \tag{3}$$

In other words, $\{d_n^{1/2}\varphi_n(i,j) \mid n \ge 1, 1 \le i,j \le d_n\}$ is an orthonormal system.

To define randomly Sidon
to replace the random $\pm$
we may use (equivalently)
either
independent random orthogonal matrices (in $O(d_n)$)
or
unitaries (in $U(d_n)$)
or
Gaussian random matrices:
Let g_n be an independent sequence of random $d_n \times d_n$-matrices, such that $\{{d_n}^{1/2} g_n(i,j) \mid 1 \leq i,j \leq d_n\}$ are i.i.d. normalized $\mathbb{C}$-valued Gaussian random variables. Note $\|g_n(i,j)\|_2 = {d_n}^{-1/2}$.

We replace $\sum \pm a_n \varphi_n$ by :

$$\sum d_n \mathrm{tr}(a_n g_n \varphi_n)$$

The definition of $\dot{\otimes}^k$-**Sidon** now means that the family of *matrix products* $(\varphi_n(t_1)\cdots\varphi_n(t_k))$ is Sidon on $(T,m)^{\otimes^k}$

Theorem (3)

Theorems (1) and (2) are still valid with the corresponding assumptions:

- *subGaussian implies* $\dot{\otimes}^2$*-Sidon*
- *randomly Sidon implies* $\dot{\otimes}^4$*-Sidon*

$$\dim(H) < \infty \quad t \mapsto \psi_1(t) \in B(H) \quad t \mapsto \psi_2(t) \in B(H)$$

$$(\psi_1 \dot{\otimes} \psi_2)(t_1,t_2) = \psi_1(t_1)\psi_2(t_2)$$

Corollary (The union problem)

The union of two "orthogonal" Sidon sets is $\dot{\otimes}^4$*-Sidon*

Example of application

Let $\chi \geq 1$ be a constant. Let T_n be the set of $n \times n$-matrices $a = [a_{ij}]$ with $a_{ij} = \pm 1/\sqrt{n}$. Let

$$A_n^\chi = \{a \in T_n \mid \|a\| \leq \chi\}.$$

This set **includes** the famous Hadamard matrices. We have then

Corollary

There is a numerical $\chi \geq 1$ such that for some C we have

$$\forall n \geq 1 \; \forall x \in M_n \quad \mathrm{tr}|x| \leq C \sup_{a', a'' \in A_n^\chi} |\mathrm{tr}(xa'a'')|.$$

Equivalently, denoting $A_n^\chi A_n^\chi = \{a'a'' \mid a', a'' \in A_n^\chi\}$ its absolutely convex hull satisfies

$$(\chi)^2 absconv[A_n^\chi A_n^\chi] \subset B_{M_n} \subset C absconv[A_n^\chi A_n^\chi]$$

Thank you !