A note on quasinormal operators

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Definitions of quasinormality-bounded case

Quasinormal operators ($AA^*A = A^*AA$), which were introduced by A. Brown (1953), form a class of operators which is properly larger than that of normal operators ($A^*A = AA^*$), and properly smaller than that of subnormal operators (A has a normal extension)

Definitions of quasinormality-unbounded case

Kaufman's definition of quasinormality

We say that a closed densely defined operator C in $\mathcal{H}$ is quasinormal if $C(C^*C) = (C^*C)C$

J. Stochel, F. H. Szafraniec definition of quasinormality

A closed densely defined operator C in $\mathcal{H}$ is quasinormal if and only if $U|C| \subset |C|U$, where $C = U|C|$ is the polar decomposition of C

- Z. J. Jablonski, I. B. Jung, J. Stochel proved that this defnitions are equivalent.

Composition operators in L^2-spaces

- $(X, \mathcal{A}, \mu)$ is a σ-finite measure space
- $\phi : X \to X$ is an $\mathcal{A}$-measurable transformation, i.e., $\phi^{-1}(\Delta) \in \mathcal{A}$ for every $\Delta \in \mathcal{A}$
- If the measure $\mu \circ \phi^{-1}$ given by $\mu \circ \phi^{-1}(\Delta) = \mu(\phi^{-1}(\Delta))$ for $\Delta \in \mathcal{A}$ is absolutely continuos with respect to μ (we say that μ is **nonsingular**), then the operator C_ϕ in $L^2(\mu)$ given by
  $\mathcal{D}(C_\phi) = \{f \in L^2(\mu) : f \circ \phi \in L^2(\mu)\}$,
  $C_\phi f = f \circ \phi, f \in \mathcal{D}(C_\phi)$
  is well-defined
- We call it a **composition** operator with **symbol** ϕ

Weighted shifts on directed trees

- Let $\mathcal{T} = (V; E)$ be a directed tree (V and E are the sets of vertices and edges of $\mathcal{T}$, respectively)
- Let $\ell^2(V)$ be the Hilbert space of square summable complex functions on V equipped with the standard inner product
- For $u \in V$, we define $e_u \in \ell^2(V)$ to be the characteristic function of the one-point set $\{u\}$.

- For a family $\lambda = \{\lambda_v\}_{v \in V^\circ}$ let us define the operator S_λ in $\ell^2(V)$ by

$$\mathcal{D}(S_\lambda) = \{f \in \ell^2(V) : \Lambda_{\mathcal{T}} f \in \ell^2(V)\}$$

$$S_\lambda f = \Lambda_{\mathcal{T}} f \quad \text{for} \quad f \in \mathcal{D}(S_\lambda);$$

- where $\Lambda_{\mathcal{T}}$ is define on a function $f : V \to \mathbb{C}$

$$(\Lambda_{\mathcal{T}} f)(v) = \begin{cases} \lambda_v f(par(v)) & \text{if } v \in V^\circ, \\ 0 & \text{otherwise.} \end{cases}$$

Characterization of quasinormal operators

Theorem

Let C be a closed densely defined operator in $\mathcal{H}$. Then the following conditions are equivalent:

- C is quasinormal
- $C^{*n}C^n = (C^*C)^n$ for every $n \in \mathbb{Z}_+$,
- there exists a (unique) spectral Borel measure E on $\mathbb{R}_+$ such that
  $C^{*n}C^n = \int_{\mathbb{R}_+} x^n E(dx)$ for $n \in \{1, 2, 3\}$
- $C^{*n}C^n = (C^*C)^n$ for every $n \in \{2, 3\}$

A question

- Is the equality $C^{*n}C^n = (C^*C)^n$ sufficient for quasinormality of a composition operator C in L^2-space?

Theorem

Let $k \geq 2$. Then any bounded injective bilateral weighted shift W that satisfies the equality $(W^*W)^k = (W^*)^k W^k$ is quasinormal.

Theorem

For every integer $n \geq 2$, there exist an injective, non-quasinormal composition operator C in L^2-space over a σ-finite measure such that

$$(C^*C)^n = C^{*n}C^n \qquad (C^*C)^k \neq C^{*k}C^k$$

for all $k \in \{2, 3, ...\} \setminus \{n\}$.

Special directed tree

Leafless and rootless directed trees with one branching vertex of valency $\aleph_0$

Let $\mathcal{T} = (V, E)$ be a directed tree with

$$V = \{-k : k \in \mathbb{Z}_+\} \sqcup \{(i,j) : i,j \in \mathbb{N}\} \tag{1}$$

and

$$E = \{(-k, -k+1) : k \in \mathbb{N}\} \sqcup \{(0,(i,1)) : i \in \mathbb{N}\} \tag{2}$$

$$\sqcup\{((i,j),(i,j+1)) : i,j \in \mathbb{N}\}.$$

(the symbol "$\sqcup$" detonates disjoint union of sets).

Theorem (Z. J .Jablonski, I. B. Jung, J. Stochel)

Let S_λ be a weighted shift on a rootless directed tree $\mathcal{T} = (V; E)$ with positive weights. Then S_λ is unitarily equivalent to a composition operator C in an L^2-space over a σ-finite measure space. Morover, if the directed tree is leafless, then C can be made injective.

$\mathcal{M}_k$ notation

We say operator A satisfies the condition $\mathcal{M}_k$ if the equality $A^{*k}A^k = (A^*A)^k$ holds.

Lemma

Let A be a bounded operator in $\mathcal{H}$ and $p, q \in \mathbb{N}$.

(i) A satisfies $\mathcal{M}_p$,$\mathcal{M}_q$, $\mathcal{M}_{p+q}$, $\mathcal{M}_{2p}$, $\mathcal{M}_{2p+q}$.

(ii) A satisfies $A^{p*}A^pA^q = A^qA^{p*}A^p$ and $\mathcal{M}_p$.

(iii) A satisfies $A^*AA^q = A^qA^*A$

Then the implication $(i) \Rightarrow (ii) \Rightarrow (iii)$ holds.

By (i), we have

$$
\begin{aligned}
A^{*(p+q)}A^pA^{*p}A^{p+q} &= A^{*q}(A^{*p}A^p)^2A^q = A^{*q}(A^*A)^{2p}A^q \\
&= A^{*q}(A^{*2p}A^{2p})A^q = A^{*2p+q}A^{2p+q} = (A^*A)^{2p+q}
\end{aligned}
$$

$$
\begin{aligned}
A^{*p}A^pA^{*(p+q)}A^{p+q} &= (A^*A)^p(A^*A)^{p+q} = (A^*A)^{2p+q} \text{ and} \\
A^{*p}A^pA^{*q}A^qA^{*p}A^p &= (A^*A)^p(A^*A)^q(A^*A)^p = (A^*A)^{2p+q}
\end{aligned}
$$

This equalities yields

$$
\begin{aligned}
&\|(A^{*p}A^{p+q} - A^qA^{*p}A^p)f\|^2 = \\
&= \langle A^{*p}A^{p+q}f, A^{*p}A^{p+q}f\rangle - 2\mathrm{Re}\langle A^{*p}A^{p+q}f, A^qA^{*p}A^pf\rangle \\
&+ \langle A^qA^{*p}A^pf, A^qA^{*p}A^pf\rangle \\
&= \langle A^{*(p+q)}A^pA^{*p}A^{p+q}f, f\rangle - 2\mathrm{Re}\langle A^{*p}A^pA^{*q}A^{*p}A^{p+q}f, f\rangle \\
&+ \langle A^{*p}A^pA^{*q}A^qA^{*p}A^pf, f\rangle = \\
&= \langle (A^*A)^{2p+q}f, f\rangle - 2\mathrm{Re}\langle A^{*p}A^pA^{*(p+q)}A^{p+q}f, f\rangle \\
&+ \langle A^{*p}A^p(A^{*q}A^q)A^{*p}A^pf, f\rangle = \\
&= \langle (A^*A)^{2p+q}f, f\rangle - 2\mathrm{Re}\langle (A^*A)^p(A^*A)^{p+q}f, f\rangle \\
&+ \langle (A^*A)^p(A^*A)^q(A^*A)^pf, f\rangle = 0
\end{aligned}
$$

Theorem

Let A be a bounded operator in $\mathcal{H}$ and $k \in \mathbb{N}$. Then the following conditions are equivalent:

(i) A is quasinormal.

(ii) A satisfies $\mathcal{M}_k$, $\mathcal{M}_{k+1}$, $\mathcal{M}_{2k}$, $\mathcal{M}_{2k+1}$.

(iii) A satisfies $\mathcal{M}_2$, $\mathcal{M}_k$, $\mathcal{M}_{k+1}$, $\mathcal{M}_{k+2}$.

(iv) A satisfies $\mathcal{M}_k$, $\mathcal{M}_{2k}$, $\mathcal{M}_{3k}$.

(iii)⇒(ii) By Lemma applied to $p = 1$ and $q = k$, we have

$$(A^*A)A^k = A^k(A^*A)$$

This equation and conditions $\mathcal{M}_k$ and $\mathcal{M}_{k+1}$ implies that

$$\begin{aligned} A^{*2k+s}A^{2k+s} &= A^{*k}A^{*k+s}A^{k+s}A^k \overset{\mathcal{M}_{k+s}}{=} A^{*k}(A^*A)^{k+s}A^k \\ &= A^{*k}A^k(A^*A)^{k+s} \overset{\mathcal{M}_k}{=} (A^*A)^{2k+s} \end{aligned}$$

for $s = 0, 1$.
Hence operator A satisfies conditions $\mathcal{M}_k$, $\mathcal{M}_{k+1}$, $\mathcal{M}_{2k}$, $\mathcal{M}_{2k+1}$. This completes the proof.

(iv)⇒(ii) We have the following equalities for $s = 2, 3$

$$(A^k)^{*s}(A^k)^s = A^{*sk}A^{sk} \overset{\mathcal{M}_{sk}}{=} (A^*A)^{sk} \overset{\mathcal{M}_k}{=} (A^{*k}A^k)^s$$

By this is equivalent to

$$(A^k)^{*s}(A^k)^s = (A^{*k}A^k)^s$$

This equalities implies that operator A^k is quasinormal, hence

$$(A^{*k}A^k)A^k = A^k(A^{*k}A^k)$$

This fact combined with condition $\mathcal{M}_k$ gives

$$(A^*A)^k A^k = A^k(A^*A)^k \tag{3}$$

which is equivalent to

$$(A^*A)A^k = A^k(A^*A)$$

This and conditions $\mathcal{M}_k$ and $\mathcal{M}_{2k}$ implies that

$$\begin{aligned} A^{*sk+1}A^{sk+1} &= A^{*sk}(A^*A)A^{sk} = A^{*sk}A^{sk}(A^*A) \\ &\overset{M_{sk}}{=} (A^*A)^{sk}(A^*A) = (A^*A)^{sk+1} \end{aligned}$$

for $s = 1, 2$.

Lemma

Assume that $M, N, T \in \mathbf{B}(\mathcal{H})$, M and N are positive, and

$$TM^k = N^k T \tag{4}$$

Then $TM = NT$

An application of Berberian's trick concerning 2×2 operator matrices gives the following equation which is equivalent to equation (4)

$$\begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}^k = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}^k \begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}$$

The Spectral Theorem shows that if $T \in \mathbf{B}(\mathcal{H})$, $T \geq 0$ and $ST^k = T^k S$, then $ST = TS$ hence

$$\begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}$$

The last equality is equivalent to $TM = NT$. This completes the proof.

Theorem

Let A be a bounded operator in $\mathcal{H}$ and $k \in \mathbb{N}$. Then the following conditions are equivalent:

(i) operators A and A^* satisfies conditions $\mathcal{M}_k$ and $\mathcal{M}_{k+1}$

(ii) A is normal.

It is clear that

$$A^k(A^{*k}A^k) = (A^kA^{*k})A^k$$

We use now, the condition $\mathcal{M}_k$ and obtain the equality

$$A^k(A^*A)^k = (AA^*)^kA^k, \tag{5}$$

which is equivalent to

$$A^k(A^*A) = (AA^*)A^k, \tag{6}$$

The same reasoning gives us

$$A^{k+1}(A^*A) = (AA^*)A^{k+1}, \tag{7}$$

An induction argument shows that for every natural number j the equality holds

$$(AA^*)^jA^{k+1} = A^{k+1}(A^*A)^j$$

In particular for $j = k$ we have

$$(AA^*)^kA^{k+1} = A^{k+1}(A^*A)^k$$

Using algebraic manipulations the last equation and conditions $\mathcal{M}_k$ and $\mathcal{M}_{k+1}$ for operator A we obtain

$$
\begin{aligned}
&\|(A^{*k}A^{k+1} - AA^{*k}A^k)f\|^2 = \\
&= \langle A^{*k}A^{k+1}f, A^{*k}A^{k+1}f\rangle - 2\Re\langle A^{*k}A^{k+1}f, AA^{*k}A^kf\rangle \\
&+ \langle AA^{*k}A^kf, AA^{*k}A^kf\rangle \\
&= \langle A^{*(k+1)}(A^kA^{*k}A^{k+1})f, f\rangle - 2\Re\langle A^{*k}A^kA^*A^{*k}A^{k+1}f, f\rangle \\
&+ \langle A^{*k}A^kA^*AA^{*k}A^kf, f\rangle = \\
&= \langle A^{*(k+1)}(A^{k+1}A^{*k}A^k)f, f\rangle - 2\Re\langle A^{*k}A^kA^{*(k+1)}A^{k+1}f, f\rangle \\
&+ \langle A^{*k}A^k(A^*A)A^{*k}A^kf, f\rangle = \\
&= \langle (A^*A)^{k+1}(A^*A)^kf, f\rangle - 2\Re\langle (A^*A)^k(A^*A)^{k+1}f, f\rangle \\
&+ \langle (A^*A)^k(A^*A)(A^*A)^kf, f\rangle = 0
\end{aligned}
$$

Hence,

$$A^{*k}A^{k+1} = AA^{*k}A^k$$

Using condition $\mathcal{M}_k$ and Lemma, we have

$$A^*A^2 = AA^*A$$

we see that A is quasinormal. The same reasoning gives us that also A^* is quasinormal